PERTURBATION OF AN EIGEN-VALUE FROM A DENSE POINT SPECTRUM: A GENERAL FLOQUET HAMILTONIAN

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Abstract. We consider a perturbed Floquet Hamiltonian $-i\partial_t + H + \beta V(\omega t)$ in the Hilbert space $L^2([0,T],\mathcal{H},dt)$. Here H is a self-adjoint operator in $\mathcal{H}$ with a discrete spectrum obeying a growing gap condition, $V(t)$ is a symmetric bounded operator in $\mathcal{H}$ depending on t 2π-periodically, $\omega = 2\pi/T$ is a frequency and β is a coupling constant. The spectrum $\mathrm{Spec}(-i\partial_t + H)$ of the unperturbed part is pure point and dense in $\mathbb{R}$ for almost every ω. This fact excludes application of the regular perturbation theory. Nevertheless we show, for almost all ω and provided $V(t)$ is sufficiently smooth, that the perturbation theory still makes sense, however, with two modifications. First, the coupling constant is restricted to a set I which need not be an interval but 0 is still a point of density of I. Second, the Rayleigh-Schrodinger series are asymptotic to the perturbed eigen-value and the perturbed eigen-vector.

CPT-97/P.3559
November 18, 1997
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1. Introduction

The so called Floquet Hamiltonians were introduced by Howland [10] and Yajima [24] in order to study time-dependent quantum systems described by an Hamilton operator $H(t)$ acting in a Hilbert space $\mathcal{H}$. Already before this strictly mathematical setting of the problem one could meet similar ideas in the physical literature [21]. In our paper we restrict ourselves to T-periodic time-dependent Hamiltonians. In this case the Floquet operator is formally written as $K = -i\partial_t + H(t)$ and it acts in the Hilbert space $\mathcal{K} = L^2([\,0,T\,],\mathcal{H},dt)$. Usually $H(t)$ is decomposed into a sum of a time-independent part H and a time-dependent perturbation $\beta V(\omega t)$ where $\omega = 2\pi/T$ and β is a parameter (coupling constant). The primary question to be answered is that of the character of the spectrum of K [9]. What makes this task difficult is the fact that, in many interesting situations, the spectrum of the Floquet Hamiltonian associated to the unperturbed (time-independent) Hamiltonian H is pure-point and dense in $\mathbb{R}$. Particularly this excludes application of the regular perturbation theory due to Rellich [20] and Kato [13]. Let us mention a few landmarks (but definitely not all of them) in the comparatively short history of the problem which have motivated us to deal with this subject.

In the article [2] Bellissard introduced a technique to study time-dependent Schrödinger equations which was inspired by the method of the proof of the classical KAM theorem [14, 1, 16]. He considered a model on the circle (in which $\mathcal{H} = L^2(S^1)$ and $H = -\Delta$ with periodic boundary conditions) and he looked for sufficient conditions to get pure-point spectrum of the associated Floquet Hamiltonian. The density of the unperturbed spectrum leads to a small divisors problem which was mastered in this paper, for appropriate diophantine frequencies ω and V's small enough, by a method similar to the original KAM algorithm. We note that Bellissard considered a perturbation V acting as a multiplication operator by a function analytic both in the time and in the spatial variable.

Soon after Combescure addressed in [5] the same question, with H being the one-dimensional harmonic oscillator and V not necessarily analytic. To cope with the lack of analyticity she has adapted the Nash-Moser regularization trick [16]. However she had to face a more severe problem: the spectrum of H did not satisfy a growing gap condition (this is an important technical property which was satisfied in the Bellissard's model). This is why she had to restrict the class of admissible perturbations, particularly excluding realistic local potentials. Let us mention also the work [3] devoted to an interesting model with constant gaps in the spectrum of H and with an analytic perturbation V.

Later on, the first two authors of the present paper considered in [6] the same question in a more abstract situation: H is discrete, simple, with a growing gap condition (see formula (2.1)), acting in a separable Hilbert space $\mathcal{H}$ and with V being not necessarily analytic. More precisely, one didn't require that the matrix entries of V in the eigen-basis of $-i\partial_t + H$ were exponentially decaying. The paper was based on a combination of two methods: the Nash-Moser trick and the adiabatic regularization due to Howland [11]. The latter method makes it possible, roughly speaking, to convert the regularity of V in the time variable into a regularity in the spatial variable. For further development of this procedure the reader can consult [17, 12]. We note that in the reference [11] Howland proposed another way to prove the pure-point character of a spectrum which was based on a "randomization" of the original operator but he failed to extend this results to the case when H was a

Schrödinger operator.

Two main characteristics are common to all the above works. First, the results are global in the sense that they describe the character of the full spectrum. Second, all these approaches are based on the accelerated convergence method which is of iterative nature. In fact, this method is an adaptation of a procedure used in the celebrated KAM result concerning perturbations of classical integrable systems. The present paper has another goal and an essentially different method was necessary to reach it. Here we concentrate on one single eigen-value. More precisely, for operators of the same type as in [6] we shall answer affirmatively the question: *Is it possible to show that one single unperturbed eigen-value gives rise to an eigen-value of the perturbed operator?*. We shall do it using a direct method, this is to say, by showing directly that the standard eigen-value equation has a solution at least for appropriate values of the coupling constant β.

In our approach the eigen-vector is written in a form of an infinite series and to verify its convergence we again have to cope with the small divisors problem (see equation (3.3)). However we don't use any kind of iterative methods and instead we rearrange partially the series and estimate its summands directly. This compensation method was probably more explicit in our previous paper [7]. This article was inspired by the pioneering work of Eliasson [8] (see also an earlier paper by Siegel [22]) and its purpose was to check some basic ideas on an explicit example. Here we treat the general case but we borrow from [7] some intermediate results, particularly this concerns Proposition 3.1 below. Apart of the rearrangement of the series we use another crucial technical trick. This is a sort of a reduction procedure based on the observation that the eigen-values of the unperturbed Floquet Hamiltonian which may be suspected to contribute by small denominators are rather rare (see Sections 5 and 6). We note that this idea, in a bit heuristic version, already appeared in the physical literature [18].

The paper is organized as follows. The main result (Theorem 2.1) is formulated in the very beginning, i.e., in Section 2. The proof is split into several steps which are carried out in the remainder of the paper, i.e., in Sections 3-8. In fact, already after reading Section 3 one can guess about the structure of the proof. Its summary is given at the end of Section 8. The paper contains three appendices. In Appendix A we present, for the sake of completeness, a proof of the fact that the spectrum of the unperturbed Floquet Hamiltonian is dense in $\mathbb{R}$ for almost all frequencies. In Appendix B we construct an example of a perturbation for which the formal solution of the equation on eigen-values (so called Rayleigh-Schrödinger series) doesn't exist. Appendix C contains a summary of the results about Lipschitz functions that we need for our approach.

2. The problem and the result

Our goal is to study a perturbed Floquet Hamiltonian $K + \beta V$ acting in

$$\mathcal{K} := L^2([0,T], dt) \otimes \mathcal{H}$$

where $\mathcal{H}$ is a given separable Hilbert space,

$$K := -i\partial_t \otimes 1 + 1 \otimes H$$

is the unperturbed (time-independent) part and β is a coupling constant. We assume that $V(t)$ is a given 2π-periodic sufficiently smooth function with values

in the space of bounded operators $\mathcal{B}(\mathcal{H})$, and $V(t)$ is symmetric for all t. The perturbation V is naturally induced by the T-periodic function $V(\omega t)$, with $\omega := 2\pi/T$ being the frequency, and it is, of course, bounded and self-adjoint. We assume further that H is a self-adjoint operator in $\mathcal{H}$, its spectrum

$$\mathrm{Spec}(H) = \{E_k;\ k \in \mathbb{N}\}$$

is discrete, simple and obeys the gap condition

$$\inf_{k\in\mathbb{N}} \frac{E_{k+1} - E_k}{(k+1)^\alpha} \geq C_E \tag{2.1}$$

where C_E and α are strictly positive constants.

Here and everywhere in what follows we adopt the convention according to which $\mathbb{N}$ stands for the set of natural numbers starting from 1 whereas $\mathbb{Z}_+$ includes also 0.

As usual, we assume the periodic boundary conditions in time. The operator K is self-adjoint and its spectrum equals

$$\mathrm{Spec}(K) = \{F_n := \omega n_1 + E_{n_2};\ n \in \mathbb{Z} \times \mathbb{N}\}.$$

Denote by f_n, $n \in \mathbb{Z} \times \mathbb{N}$, the corresponding normalized eigen-vectors and by P_n the orthogonal projector onto $\mathbb{C}f_n$. With the help of this eigen-basis we identify the Hilbert space $\mathcal{K}$ with $l^2(\mathbb{Z} \times \mathbb{N})$ and all relevant operators with their matrices. Particularly the perturbation V is represented by the matrix (V_{mn}),

$$\begin{aligned} V_{mn} &= \frac{1}{T}\int_0^T \langle e_{m_2}, V(\omega t) e_{n_2}\rangle_{\mathcal{H}} \, \exp\big(i\omega(n_1 - m_1)t\big)\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle e_{m_2}, V(t) e_{n_2}\rangle_{\mathcal{H}} \, \exp\big(i(n_1 - m_1)t\big)\, dt\,, \end{aligned} \tag{2.2}$$

where $\{e_k;\ k \in \mathbb{N}\}$ denotes the orthonormal eigen-basis of H.

Note that the matrix entries of V don't depend on ω and so the frequency occurs only in the eigen-values of K. The problem depends on two parameters – β and also the period T. However, in the very beginning, we shall fix ω so that a diophantine condition (cf. (3.11)) is satisfied. Afterwards we don't move the value ω anymore and study the dependence only on the coupling constant.

We have just presented all the incoming data. Let us now formulate the problem. We fix once for all an index $\eta \in \mathbb{Z} \times \mathbb{N}$ and write

$$P := P_\eta \quad \text{and} \quad Q := 1 - P.$$

Similarly, we redenote $f := f_\eta$ and $F := F_\eta$; so $Kf = Ff$ and $Pf = f$, $Qf = 0$, with $\|f\| = 1$. We ask whether the operator $K + \beta V$ possesses also an eigen-value $F(\beta)$ which could be regarded as being inherited from the eigen-value F of K. The regular perturbation theory due to Rellich [20] and Kato [13] in no way provides an answer to this question since the set $\mathrm{Spec}(K) = \omega\mathbb{Z} + \mathrm{Spec}(H)$ is dense in $\mathbb{R}$ for almost all $\omega > 0$. This property of the spectrum is quite familiar, nonetheless we present an elementary proof in Appendix A. Recall that the basic assumption for the regular theory to go through is that the eigen-value F is isolated. Also

because of the density of the spectrum, it makes little sense trying to relate, for a single value of the parameter β, an eigen-value $F(\beta)$ of $K+\beta V$ to the distinguished eigen-value F of K. But we shall show that it is reasonable to relate to F a whole function $F(\beta)$, for β running over some domain in the vicinity of zero.

In our case, F can be an accumulation point of $\mathrm{Spec}(K)$. On the other hand, F is a simple eigen-value for a generic ω and so the operator $K-F$ is injective on the subspace $\mathrm{Ran}(Q)$. In fact, practically all subsequent manipulations will be concerned with this subspace while the vector f plays a role of a "source". This is reflected in the notation; for an operator X in $\mathcal{K}$ we denote by $\hat{X}$ its block corresponding to the subspace $\mathrm{Ran}(Q)$:

$$\hat{X} := QXQ \quad \text{as an operator in } \mathrm{Ran}(Q). \tag{2.3}$$

Then $(\hat{K}-F)^{-1}$ is a self-adjoint possibly unbounded operator.

There are more distinctions when comparing with the regular case. We will discuss this point in a bit more detail in Section 3. Here we recall that, according to the Rellich-Kato theorem – the basic result of the regular perturbation theory, if the eigen-value F was simple and isolated then $F(\beta)$ would be an analytic function on a neighbourhood of the origin. The same remark applies to the eigen-vector $f(\beta)$ provided a convenient normalizing condition has been imposed making it unambiguous. For example, a normalization frequent in the physical literature [15] requires that

$$\langle f, f(\beta)\rangle = 1 \Longleftrightarrow f(\beta) - f \in \mathrm{Ran}(Q) \tag{2.4}$$

is valid for all β from the corresponding domain. The analytic functions

$$\begin{aligned} F(\beta) &= F + \beta\lambda_1 + \beta^2\lambda_2 + \dots, \\ f(\beta) &= f + \beta g_1 + \beta^2 g_2 + \dots, \end{aligned}$$

are known as the Rayleigh-Schrödinger (RS) series, with the coefficients $\lambda_j \in \mathbb{R}$ and $g_j \in \mathrm{Ran}(Q)$ expressed explicitly [13, 19]. More details are given in Section 4. Here we recall only that

$$\lambda_1 = \langle f, Vf\rangle, \ \lambda_2 = -\langle QVf, (\hat{K}-F)^{-1}QVf\rangle. \tag{2.5}$$

On the contrary, this seems to be an intrinsic feature for the problems with dense point spectrum that the common domain I for the functions $F(\beta)$ and $f(\beta)$ cannot be chosen as an interval. Because of the resonance effects it possesses numerous "holes". Nevertheless 0 can be a point of density of I. Furthermore, the relation of the RS series to the functions $F(\beta)$ and $f(\beta)$ is not straightforward. A priori it is even not clear whether the coefficients λ_j and g_j are well defined. For example, the existence of λ_2 in (2.5) is guaranteed by the condition $QVf \in \mathrm{Dom}((\hat{K}-F)^{-1})$ which is not obvious at all. Fortunately it turns out that the coefficients do exist, up to some order, provided $V(t)$ is sufficiently smooth. Then the RS series don't determine $F(\beta)$ and $f(\beta)$ directly but instead they describe the asymptotic behaviour of these functions as $\beta \to 0$.

Now we are ready to formulate the result. Here $|X|$ stands for the Lebesgue measure of a measurable set X.

Theorem 2.1. *Suppose that a self-adjoint operator H with a discrete spectrum obeys the gap condition (2.1) and a symmetric operator-valued function $V(t) \in C^r$ in the strong sense, with $r \geq 2$ and $r > 16/\alpha$. Then there exists a set $\Omega \subset]0, +\infty[$ of full Lebesgue measure such that, for all $\omega \in \Omega$ and any $\eta \in \mathbb{Z} \times \mathbb{N}$ fixed, the Rayleigh-Schrödinger coefficients $\lambda_j \in \mathbb{R}$ and $g_j \in \mathrm{Ran}(Q)$, $1 \leq j \leq \ell$, are well defined, with ℓ being the greatest integer which fulfills*

$$\ell < \frac{r\alpha}{4} - 2. \tag{2.6}$$

If, moreover, the second coefficient $\lambda_2 \neq 0$ (as given in (2.5)) then there exist a real function $F(\beta)$ and a $\mathcal{K}$-valued function $f(\beta)$ defined on a common domain I and having the properties:

(1) $f(\beta) \in \mathrm{Dom}(K)$, $\langle f, f(\beta)\rangle = 1$, *and* $(K + \beta V)f(\beta) = F(\beta)f(\beta)$ *for all* $\beta \in I$,

(2) $\lim_{\delta\downarrow 0} |I \cap [-\delta, \delta]|/2\delta = 1$,

(3) $F(\beta) = F + \beta\lambda_1 + \cdots + \beta^\ell\lambda_\ell + O(|\beta|^{\ell+1})$,
$f(\beta) = f + \beta g_1 + \cdots + \beta^\ell g_\ell + O(|\beta|^{\ell+1})$.

From the construction of the set Ω (c.f. (3.10) and Proposition 3.1) it is evident that the eigen-value F of K is simple for all $\omega \in \Omega$. Furthermore, let us note that if $V(t) \in C^\infty$ then the coefficients λ_j and g_j exist for all $j \in \mathbb{N}$ and the property (3) means that the functions $F(\beta)$ and $f(\beta)$ have asymptotic expansions at $\beta = 0$ coinciding with the RS series.

We conclude this section by a brief comparison of this theorem with some previous results. This concerns, first of all, the mutual role of the two parameters ω and β. A notable approach to the spectral problem of the operator $K + \beta V$ goes back to Bellissard [2] (see also [5], [6]). Also in this case, the spectrum of the unperturbed Hamiltonian H was supposed to obey the same type of gap condition (2.1). Under some smoothness assumptions on $V(t)$, one is able to show that, for each sufficiently small β, there exists a set of "non-resonant" frequencies $\Omega(\beta)$ such that the Lebesgue measure of the complement of $\Omega(\beta)$ is reasonably small and the operator $K + \beta V$ is pure point for each $\omega \in \Omega(\beta)$. The dependence of $\Omega(\beta)$ on β is to be emphasized. On the contrary, the above theorem focuses only on one distinguished eigen-value. But in this case one can choose the set Ω independently of β so that it covers almost all frequencies $\omega > 0$ in the Lebesgue sense. The basic problem now is to construct a convenient domain I for the coupling constant β, with $\omega \in \Omega$ being fixed. Naturally I depends on the choice of the unperturbed eigen-value.

We split the proof of Theorem 2.1 into several steps, each of them treated in one of the subsequent sections. A summary of all the steps is given at the end of Section 8.

3. Projection method, comparison with the regular case

We start the proof of Theorem 2.1 from the perturbed equation on eigen-values,

$$(K + \beta V)(f + g) = (F + \lambda)(f + g), \tag{3.1}$$

with $\lambda \in \mathbb{R}$ and, according to the normalization (2.4), $g \in \mathrm{Ran}(Q)$. Applying to (3.1) the complementary projectors P and Q (commuting with K) we obtain an

equivalent set of equations (recall (2.3))

$$\lambda = \beta\langle Vf, f\rangle + \beta\langle Vf, g\rangle, \tag{3.2}$$

$$(\hat{K} + \beta\hat{V} - F - \lambda)g = -\beta QVf\,. \tag{3.3}$$

For a while we shall consider λ as another auxiliary parameter and we will try to solve the equation (3.3), referred to as the eigen-vector equation from now on. Its solution is a vector-valued function $g = g(\beta, \lambda)$ depending on both parameters β and λ, and taking values in $\mathrm{Ran}(Q)$. Plugging $g(\beta, \lambda)$ into the equality (3.2) we get an implicit equation $\lambda = G(\beta, \lambda)$ from which one should extract a function $\lambda = \lambda(\beta)$. Then

$$F(\beta) = F + \lambda(\beta) \quad \text{and} \quad f(\beta) = f + g(\beta, \lambda(\beta)) \tag{3.4}$$

will be the sought solution to our problem. This projection method was rediscovered many times in the past and bears various names: Brillouin-Wigner, Feshbach, Grushin, Schur, ...

Naturally this procedure can be applied to the regular case as well and one can rederive this way the Rellich-Kato theorem. In order to emphasize the difference between the regular and non-regular cases we sketch below the basic steps. But before doing it let us introduce some more notation used throughout the paper. Set

$$\Gamma_0 := (\hat{K} - F)^{-1}, \tag{3.5}$$

$$\Gamma_\lambda := (\hat{K} - F - \lambda)^{-1} = (1 - \lambda\Gamma_0)^{-1}\Gamma_0. \tag{3.6}$$

Thus Γ_0 is a self-adjoint operator acting in $\mathrm{Ran}(Q)$ provided F is a simple eigen-value of K. The same holds true for Γ_λ if $\lambda \notin \mathrm{Spec}(\hat{K} - F)$.

The regular case is characterized by the condition

$$\mathrm{dist}(F, \mathrm{Spec}(K) \setminus \{F\}) =: d > 0. \tag{3.7}$$

Hence the operator Γ_0 is even bounded and $\|\Gamma_0\| = d^{-1}$. Moreover, Γ_λ is bounded as well and depends analytically on λ in the domain $|\lambda| < d$. However K itself need not be bounded and one can even consider a more general situation with V being relatively bounded with respect to K. This assumption implies that $\|\Gamma_0\hat{V}\| = \|\hat{V}\Gamma_0\| < \infty$ and it is sufficient to ensure that the operator $1 + \beta\Gamma_\lambda\hat{V}$ is invertible provided the parameters β and λ belong to the domain

$$d\,\|\hat{V}\Gamma_0\|\,|\beta| + |\lambda| < d\,. \tag{3.8}$$

Consequently, there exists a unique solution to (3.3) given by

$$g(\beta, \lambda) = -\beta(1 + \beta\Gamma_\lambda\hat{V})^{-1}\Gamma_\lambda QVf\,.$$

Obviously, the function $g(\beta, \lambda)$ is analytic in the domain (3.8) and its values belong to $\mathrm{Dom}(\hat{K} - F - \lambda) \subset \mathrm{Dom}(K)$. The equality (3.2) then leads to the implicit equation

$$\begin{aligned} &\lambda = G(\beta, \lambda), \quad \text{with} \\ &G(\beta, \lambda) = \beta\langle Vf, f\rangle - \beta^2\langle QVf, (1 + \beta\Gamma_\lambda\hat{V})^{-1}QVf\rangle\,. \end{aligned}$$

Since $G(\beta,\lambda)$ is analytic and

$$(\lambda - G(\beta,\lambda))|_{(\beta,\lambda)=(0,0)} = 0, \quad \partial_\lambda(\lambda - G(\beta,\lambda))|_{(\beta,\lambda)=(0,0)} = 1\,,$$

the implicit mapping theorem tells us that there exists a unique analytic function $\lambda = \lambda(\beta)$ defined on a neighbourhood of the origin and such that $\lambda(0) = 0$, $\lambda(\beta) = G(\beta,\lambda(\beta))$. In accordance with (3.4) we get both the perturbed eigen-value $F(\beta)$ and the eigen-vector $f(\beta)$ as uniquely determined analytic functions.

Let us return to our problem with dense point spectrum and with V being a bounded perturbation. Violation of the condition (3.7) means exactly that the operator Γ_0 is unbounded. We shall need another but weaker condition in order to be still able to cope with the equation (3.3). Diophantine estimates are the standard tool used widely in this situation. Let us first introduce the relevant exponents. The integer ℓ, as specified in Theorem 2.1 (cf. (2.6)), obeys

$$\ell \geq 2 \quad \text{and} \quad 4\ell + 8 < r\alpha.$$

Hence one can find reals $\tau > 4$ and $\sigma > 1$ such that

$$\tau(\ell+2) \leq r\alpha \quad \text{and} \quad 2\sigma + 2 < \tau. \tag{3.9}$$

Next we define the set of non-resonant frequencies,

$$\Omega_\eta := \{\omega > 0;\ \inf_{n\in\mathbb{Z}\times\mathbb{N},\ n\neq\eta} n_2^\sigma |F_n - F| > 0\}. \tag{3.10}$$

A simple adaptation of the proof of Lemma 4 in [7] shows that if $\sigma > 1$ then $\Omega_\eta \subset]0,+\infty[$ is of full Lebesgue measure. It is clear that a non-resonant frequency can be even chosen for all indices η simultaneously.

Proposition 3.1. *Suppose that $\sigma > 1$. Then almost all $\omega > 0$ belong to*

$$\Omega := \bigcap_{m\in\mathbb{Z}\times\mathbb{N}} \Omega_m\,.$$

We fix, for the rest of the paper, a non-resonant frequency $\omega \in \Omega$. Then eigen-values of $\hat{K} - F$ fulfill the diophantine estimate

$$|F_n - F| \geq \gamma\, n_2^{-\sigma} \quad \text{for all } n \in \mathbb{Z}\times\mathbb{N},\ n \neq \eta, \tag{3.11}$$

with some constant $\gamma > 0$. In addition, the property (3.11) guarantees that F is a simple eigen-value of K. We shall write

$$\psi(k) := \gamma\, k^{-\sigma}, \quad \tilde{\psi}(k) := \frac{\gamma}{2}\, k^{-\tau}\,. \tag{3.12}$$

We would like to warn the reader that, in order to avoid introducing additional symbols, the restrictions (3.9) on τ will be applied in the subsequent procedure only at those places where they have some consequences, otherwise τ can be any real number. Similarly, ℓ can be any non-negative integer if not specified otherwise.

Let us finish shortly the comparison of the regular and non-regular cases by indicating some forthcoming steps. The discrete function $\tilde{\psi}(k)$ given in (3.12) will

be used later, in Section 6, in another diophantine estimate involving the parameters β and λ and defining a closed set $\mathcal{D} \subset \mathbb{R}^2$. We shall be able to solve the eigen-vector equation (3.3) provided $(\beta, \lambda) \in \mathcal{D}$ getting this way a vector-valued function $g(\beta, \lambda)$. Consequently the function $G(\beta, \lambda) := \beta \langle Vf, g(\beta, \lambda) \rangle$ is defined only on the set $\mathcal{D}$, too, but fortunately one can show that G belongs to the Lipschitz class $\mathrm{Lip}(\ell+1, \mathcal{D})$, with ℓ specified in Theorem 2.1. This enables one to apply the Whitney extension theorem in order to extend G from $\mathcal{D}$ to $\mathbb{R}^2$. Making the standard simplifying assumption that $\langle Vf, f \rangle = 0$ one again arrives at the implicit equation $\lambda = \tilde{G}(\beta, \lambda)$, with the extended right hand side. The implicit mapping theorem guarantees the existence of a solution $\lambda = \tilde{\lambda}(\beta)$. However one has to restrict the function $\tilde{\lambda}$ to the set I determined by the condition $(\beta, \tilde{\lambda}(\beta)) \in \mathcal{D}$. Thus the resulting function $\lambda(\beta)$ is not defined on an interval but, on the other hand, one can verify that its domain I is still reasonably dense at the origin.

4. Perturbation series

In this section we summarize a few basic facts about the RS series, particularly we recall the explicit expressions for coefficients in a form relying on some combinatorial notions. Basically we adopt the physical point of view according to which one seeks the eigen-vector $f(\beta)$ normalized by $\langle f, f(\beta) \rangle = 1$ [15]. In a more mathematically oriented approach one prefers to treat the orthogonal projector $P(\beta)$ onto the 1-dimensional subspace $\mathbb{C}f(\beta)$ rather than the vector $f(\beta)$ itself. Then the corresponding formulas take an optically different form [13]. But, of course, our choice is only a matter of taste and convenience as the both approaches are obviously equivalent; for example

$$f(\beta) = \frac{1}{\langle f, P(\beta)f \rangle} P(\beta)f \,.$$

On the other hand, the eigen-value $F(\beta)$ is unambiguous and the result must be the same in any case. This point has been discussed shortly in [7].

We are forced to use a bit more general setting since the functions $F(\beta)$ and $f(\beta)$ need not be analytic and instead they are characterized by their asymptotics. However this doesn't cause a serious complication.

Lemma 4.1. *Suppose that 0 is an accumulation point of a closed set $I \subset \mathbb{R}$, $\ell \in \mathbb{N}$, and we are given a real function $F(\beta)$ and a $\mathcal{K}$-valued function $f(\beta)$, both defined on I and having asymptotics at $\beta = 0$:*

$$F(\beta) = F + \beta\lambda_1 + \cdots + \beta^\ell \lambda_\ell + O(|\beta|^{\ell+1}), \tag{4.1}$$

$$f(\beta) = f + \beta g_1 + \cdots + \beta^\ell g_\ell + O(|\beta|^{\ell+1}). \tag{4.2}$$

Suppose, moreover, that for all $\beta \in I$, $f(\beta) \in \mathrm{Dom}(K)$ and

$$(K + \beta V)f(\beta) = F(\beta)f(\beta) \,. \tag{4.3}$$

Then $f, g_1 \ldots, g_\ell \in \mathrm{Dom}(K)$ and

$$Kf(\beta) = Kf + \beta\, Kg_1 + \cdots + \beta^\ell\, Kg_\ell + O(|\beta|^{\ell+1}). \tag{4.4}$$

Proof. The function $Kf(\beta)$ has an asymptotic as well since

$$Kf(\beta) = -\beta\, Vf(\beta) + F(\beta)f(\beta) = u_0 + \beta u_1 + \cdots + \beta^\ell u_\ell + O(|\beta|^{\ell+1}).$$

Redenote temporarily f as g_0. Proceeding by induction in j we shall show that $g_j \in \mathrm{Dom}(K)$ and $Kg_j = u_j$, $j = 0, 1, \ldots, \ell$. This is obvious for $j = 0$ as $g_0 = f(0)$ and $Kg_0 = Kf(0) = u_0$. Suppose that $j \geq 1$ and set temporarily, for $\beta \neq 0$,

$$h_j(\beta) := \beta^{-j}\left(f(\beta) - g_0 - \beta\, g_1 - \cdots - \beta^{j-1}\, g_{j-1}\right).$$

Then $h_j(\beta) \to g_j$ and, by the induction hypothesis, $h_j(\beta) \in \mathrm{Dom}(K)$ and $Kh_j(\beta) \to u_j$, as $\beta \to 0$. But K is closed and so $g_j \in \mathrm{Dom}(K)$ and $Kg_j = u_j$. $\square$

From the existence of the asymptotics (4.1), (4.2) and (4.4) follows immediately that the corresponding coefficients on the both sides of (4.3) coincide up to the order ℓ. This leads to the system of equations ($g_0 \equiv f$)

$$(K - F)g_M = -V g_{M-1} + \sum_{j=1}^{M-1} \lambda_j\, g_{M-j} + \lambda_M\, f, \quad 1 \leq M \leq \ell. \tag{4.5}$$

If $f(\beta)$ obeys the normalization (2.4), and so $g_j \in \mathrm{Ran}(Q)$ for $j \geq 1$, one can again separate the parts belonging to $\mathrm{Ran}(P)$ and $\mathrm{Ran}(Q)$ getting this way

$$\begin{aligned} &(\hat{K} - F)g_M = -\hat{V} g_{M-1} + \sum_{j=1}^{M-1} \lambda_j\, g_{M-j} \quad \text{where} \\ &\lambda_M = \langle QVf, g_{M-1}\rangle,\ M = 1, \ldots, \ell, \end{aligned} \tag{4.6}$$

(for $M = 1$, $\hat{V} g_{M-1}$ should be replaced by QVf). We still assume that $(\hat{K} - F)^{-1} = \Gamma_0$ exists. Clearly one can calculate, successively and unambiguously, the vectors $g_1, \ldots, g_\ell$, and consequently the numbers $\lambda_1, \ldots, \lambda_\ell$ as well provided one can show that $g_1, \ldots, g_{\ell-1}$ and $QVf, \hat{V} g_1, \ldots, \hat{V} g_{\ell-1}$ belong to $\mathrm{Ran}(\Gamma_0)$. In this case we can rewrite (4.6) in the form

$$g_M = -\Gamma_0 \hat{V} g_{M-1} + \sum_{j=1}^{M-1} \lambda_j\, \Gamma_0 g_{M-j}, \quad M = 1, \ldots, \ell. \tag{4.7}$$

One deduces readily from (4.7) that g_M is a linear combination of the vectors

$$\Gamma_0^{s_1} \hat{V} \ldots \hat{V} \Gamma_0^{s_p} QVf \quad \text{with } 1 \leq p \leq M,\ (s_1, \ldots, s_p) \in \mathbb{N}^p,\ \text{and } \sum s_i \leq M. \tag{4.8}$$

Hence the existence of vectors (4.8), for $M = 1, \ldots, \ell$, represents a sufficient condition for the system (4.6) to have a unique solution.

Before approaching the explicit expressions let us recall a bit of combinatorics. The set of rooted N-trees $\mathcal{T}(N) \subset \mathbb{Z}_+^N$ is characterized by the condition ($|\nu| := \nu_1 + \cdots + \nu_N$):

$$\begin{aligned} &\nu = (\nu_1, \ldots, \nu_N) \in \mathcal{T}(N) \Longleftrightarrow \\ &\nu_k + \cdots + \nu_N \leq N - k \text{ for } 2 \leq k \leq N, \text{ and } |\nu| = N - 1. \end{aligned}$$

Obviously $\nu_N = 0$, and if $N \geq 2$ then $\nu_1 \geq 1$. It is also quite easy to verify a composition rule for two trees, namely

$$\nu' \in \mathcal{T}(N'),\ \nu'' \in \mathcal{T}(N'') \Longrightarrow \nu = (\nu', \nu'') + (1, 0, \ldots, 0) \in \mathcal{T}(N' + N'').$$

As stated in the following lemma this procedure is invertible. We don't recall the proof.

Lemma 4.2. *Suppose that $\nu \in \mathcal{T}(N)$ and $N \geq 2$. Then there exists a unique decomposition $\nu = (\nu', \nu'') + (1, 0, \dots, 0)$ where $\nu' \in \mathcal{T}(N')$, $\nu'' \in \mathcal{T}(N'')$ and $N' + N'' = N$.*

Now we are ready to describe the solution to the system (4.6).

Proposition 4.3. *Suppose that the vectors $\Gamma_0^{s_1}\hat{V}\dots\Gamma_0^{s_{p-1}}\hat{V}\Gamma_0^{s_p}QVf$ are well defined for all $p \in \mathbb{N}$, $1 \leq p \leq \ell$, and all p-tuples $(s_1, \dots, s_p) \in \mathbb{N}^p$ such that $\sum s_i \leq \ell$. Then there exists a unique ℓ-tuple $g_1, \dots, g_\ell$ solving the system of equations (4.6).*

Suppose, in addition, that $\langle Vf, f\rangle = 0$. Then the solution is given by the formula ($1 \leq M \leq \ell$)

$$g_M = \sum_{N\in\mathbb{N}} \sum_{\nu\in\mathcal{T}(N)} \sum_{k(1),\dots,k(N)\in\mathbb{N}} \sum_{\mu(1)\in\mathbb{N}^{k(1)},\dots,\mu(N)\in\mathbb{N}^{k(N)}} \mathfrak{G}_M(N, \nu, k(j), \mu(j)) \tag{4.9a}$$

where the range of summation is restricted by the conditions

$$k(1) + \dots + k(N) + N = M + 1,\ |\mu(j)| = k(j) + \nu_j \quad \textit{for } 1 \leq j \leq N\,, \tag{4.9b}$$

and

$$\begin{aligned}\mathfrak{G}_M(N, \nu, k(j), \mu(j)) :=& (-1)^{M+N+1} \prod_{j=2}^{N} \langle Vf,\, \Gamma_0^{\mu(j)_1}\hat{V}\Gamma_0^{\mu(j)_2}\dots\hat{V}\Gamma_0^{\mu(j)_{k(j)}}Vf\rangle \\ &\times \Gamma_0^{\mu(1)_1}\hat{V}\Gamma_0^{\mu(1)_2}\dots\hat{V}\Gamma_0^{\mu(1)_{k(1)}}Vf\,.\end{aligned} \tag{4.9c}$$

The numbers $\lambda_1, \dots, \lambda_\ell$ are given correspondingly by $\lambda_1 = \langle Vf, f\rangle = 0$ and, for $2 \leq M \leq \ell$,

$$\begin{aligned}\lambda_M &= \langle Vf, g_{M-1}\rangle \\ &= \sum_{N\in\mathbb{N}} \sum_{\nu\in\mathcal{T}(N)} \sum_{k(1),\dots,k(N)\in\mathbb{N}} \sum_{\mu(1)\in\mathbb{N}^{k(1)},\dots,\mu(N)\in\mathbb{N}^{k(N)}} \mathfrak{L}_M(N, \nu, k(j), \mu(j))\end{aligned} \tag{4.10a}$$

where the range of summation is restricted by the conditions

$$k(1) + \dots + k(N) + N = M,\ |\mu(j)| = k(j) + \nu_j \quad \textit{for } 1 \leq j \leq N\,, \tag{4.10b}$$

and

$$\mathfrak{L}_M(N, \nu, k(j), \mu(j)) := (-1)^{M+N} \prod_{j=1}^{N} \langle Vf,\, \Gamma_0^{\mu(j)_1}\hat{V}\Gamma_0^{\mu(j)_2}\dots\hat{V}\Gamma_0^{\mu(j)_{k(j)}}Vf\rangle\,. \tag{4.10c}$$

Proof. The first part of the proposition has been discussed above. Let us show that the vectors g_M given in (4.9) obey the relation (4.7). This is easy to check for $M = 1$. Then necessarily $N = 1$ and so, as $\mathcal{T}(1) = \{(0)\}$, the formula (4.9) gives the correct answer $g_1 = -\Gamma_0 Vf$. Suppose that $M \geq 2$. Observe that the assumption $\langle Vf, f\rangle = 0$ implies that $\lambda_1 = 0$ and so the summation index on the RHS of (4.7) starts from the value $j = 2$. Moreover, $Vf \in \operatorname{Ran}(Q)$. The verification is based on the following two equalities. First,

$$-\Gamma_0\hat{V}\mathfrak{G}_{M-1}(N', \nu', k(j)', \mu(j)') = \mathfrak{G}_M(N, \nu, k(j), \mu(j)) \tag{4.11a}$$

where

$$\begin{aligned}&N=N',\ \nu=\nu'\in\mathcal{T}(N),\ k(1)=k(1)'+1,\ k(2)=k(2)',\dots,k(N)=k(N)',\\&\mu(1)=(1,\mu(1)'),\ \mu(2)=\mu(2)',\dots,\mu(N)=\mu(N)'.\end{aligned}\tag{4.11b}$$

Second, if $1\leq M'$, $2\leq M''$ and $M=M'+M''$ then

$$\mathfrak{L}_{M''}(N'',\nu'',k(j)'',\mu(j)'')\mathfrak{G}_{M'}(N',\nu',k(j)',\mu(j)')=\mathfrak{G}_M(N,\nu,k(j),\mu(j))\tag{4.12a}$$

where

$$\begin{aligned}&N=N'+N'',\ \nu=(\nu',\nu'')+(1,0,\dots,0)\in\mathcal{T}(N),\\&k(1)=k(1)',\dots,k(N')=k(N')',k(N'+1)=k(1)'',\dots,k(N'+N'')=k(N'')'',\\&\mu(1)=\mu(1)'+(1,0,\dots,0),\ \mu(2)=\mu(2)',\dots,\mu(N')=\mu(N')',\\&\mu(N'+1)=\mu(1)'',\dots,\mu(N'+N'')=\mu(N'')''.\end{aligned}\tag{4.12b}$$

On the other hand, consider a summand $\mathfrak{G}_M(N,\nu,k(j),\mu(j))$. We distinguish two cases. If $\mu(1)_1=1$ then necessarily $k(1)\geq 2$ and there exists a unique multiindex $(N',\nu',k(1)',\dots,k(N')',\mu(1)',\dots,\mu(N')')$ determining a summand $\mathfrak{G}_{M-1}$ such that (4.11) holds. If $\mu(1)_1\geq 2$ then necessarily $N\geq 2$ and, in virtue of Lemma 4.2, there exists a unique decomposition $\nu=(\nu',\nu'')+(1,0,\dots,0)$ where $\nu'\in\mathcal{T}(N')$, $\nu''\in\mathcal{T}(N'')$ and $N'+N''=N$. Set

$$M'=k(1)+\dots+k(N')+N'-1,\ M''=k(N'+1)+\dots+k(N)+N''.$$

Observe that $N''\geq 1$ implies $M''\geq 2$. This way one obtains unambiguously two multiindices $(N',\nu',k(j)',\mu(j)')$ and $(N'',\nu'',k(j)'',\mu(j)'')$ determining respectively summands $\mathfrak{G}_{M'}$ and $\mathfrak{L}_{M''}$ such that (4.12) holds. This completes the verification. □

5. Set of critical indices, existence of the Rayleigh-Schrödinger coefficients

Let us continue the proof of Theorem 2.1. The arbitrarily small numbers in $\operatorname{Spec}(\hat{K}-F)$, so called small denominators, represent the principal difficulty we have encountered in the preceding discussion. This is why the operator $\Gamma_0=(\hat{K}-F)^{-1}$ is not bounded and thus it is not a priori clear whether the assumptions of Proposition 4.3 are fulfilled and whether the RS coefficients exist at all. The second basic ingredient of our approach, apart of the projection method, is the observation that the indices suspected of enumerating small denominators are distributed rather rarely in the lattice $\mathbb{Z}\times\mathbb{N}$. We introduce the set $\mathcal{S}\subset\mathbb{Z}\times\mathbb{N}\setminus\{\eta\}$ of "critical" indices by imposing the condition

$$n\in\mathcal{S}\Longleftrightarrow F_n-F\in\left]-\frac{\omega}{2},\frac{\omega}{2}\right].\tag{5.1}$$

Clearly, to each $n_2\in\mathbb{N}$, $n_2\neq\eta_2$, there exists exactly one $n_1\in\mathbb{Z}$ such that $(n_1,n_2)\in\mathcal{S}$ and there is no such n_1 for $n_2=\eta_2$. In other words, the projection $\mathcal{S}\to\mathbb{N}\setminus\{\eta_2\}:n\mapsto n_2$ is one-to-one. Roughly speaking, the indices from the set $\mathcal{S}$ are situated closely to the curve $n_1=\eta_1+(E_{\eta_2}-E_{n_2})/\omega$.

Now the gap condition (2.1) can be employed to get more information about the set $\mathcal{S}$. It is quite useful to observe that another inequality follows straightforwardly from (2.1), namely

$$|E_j - E_k| \geq \frac{C_E}{1+\alpha}\, |j-k|\, \max\{j^\alpha, k^\alpha\}, \quad \forall j,k \in \mathbb{N}. \tag{5.2}$$

Indeed, if $j > k$ then

$$\begin{aligned} E_j - E_k &= \sum_{p=k}^{j-1} (E_{p+1} - E_p) \geq C_E \int_k^j s^\alpha\, ds \\ &= \frac{C_E}{1+\alpha}\, (j^{1+\alpha} - k^{1+\alpha}) \geq \frac{C_E}{1+\alpha}\, (j-k)\, j^\alpha\,. \end{aligned}$$

Using (5.1) one derives that, for $m,n \in \mathcal{S}$,

$$|m_1 - n_1| = \frac{1}{\omega}\, |F_m - F_n - E_{m_2} + E_{n_2}| \geq \frac{1}{\omega}\, |E_{m_2} - E_{n_2}| - 1\,. \tag{5.3}$$

A combination of (5.3) and (5.2) yields

$$m,n \in \mathcal{S} \Longrightarrow 1 + |m_1 - n_1| \geq \frac{C_E}{\omega(1+\alpha)}\, \max\{m_2^\alpha, n_2^\alpha\}\, |m_2 - n_2|\,. \tag{5.4}$$

Similarly,

$$m \in \mathcal{S} \Longrightarrow 1 + |m_1 - \eta_1| > \frac{C_E}{\omega(1+\alpha)}\, \max\{m_2^\alpha, \eta_2^\alpha\}\, |m_2 - \eta_2|\,. \tag{5.5}$$

The set $\mathcal{S}$ induces a splitting of the subspace $\mathrm{Ran}(Q)$ into the "singular" and "regular" parts. This idea will be exploited more systematically in Section 6. Here we introduce the corresponding projectors,

$$P_S := \sum_{n\in\mathcal{S}} P_n, \quad P_R := Q - P_S\,.$$

Note that

$$\|(\hat{K} - F)P_S\| \leq \frac{\omega}{2}, \quad \|\Gamma_0 P_R\| \leq \frac{2}{\omega}\,. \tag{5.6}$$

Hence the restriction of Γ_0 to the subspace $\mathrm{Ran}(P_R)$ is quite harmless.

Let us switch to the problem of RS coefficients. To show their existence, and also later in Section 6, we shall need an inequality with commutators. First we specify the underlying notions. Let A be a closed, densely defined operator in $\mathcal{K}$ and $X \in \mathcal{B}(\mathcal{K})$. By saying that $\mathrm{ad}_A X$ is bounded we mean that: $\mathrm{Dom}(A) \subset \mathrm{Dom}(AX)$ and the operator $AX - XA$ is bounded on $\mathrm{Dom}(A)$, and so it can be unambiguously extended to an operator from $\mathcal{B}(\mathcal{K})$ that we call $\mathrm{ad}_A X$. Particularly, $[\,A, X\,] = 0$ is equivalent to: $\mathrm{Dom}(A) \subset \mathrm{Dom}(AX)$ and $AX = XA$ on $\mathrm{Dom}(A)$. One has the Leibniz rule in the following sense: if $X_1, X_2 \in \mathcal{B}(\mathcal{K})$ and both $\mathrm{ad}_A X_1,\ \mathrm{ad}_A X_2$ are bounded then so is $\mathrm{ad}_A(X_1X_2)$ and it holds

$$\mathrm{ad}_A(X_1X_2) = (\mathrm{ad}_A X_1)X_2 + X_1(\mathrm{ad}_A X_2).$$

More generally, saying that $\mathrm{ad}_A^r X$ is bounded, with $r \in \mathbb{Z}_+$, means that: $\mathrm{Dom}(A^r)$ is dense in $\mathcal{K}$, $\mathrm{Dom}(A^j) \subset \mathrm{Dom}(A^jX)$ for all $j,\ 0 \leq j \leq r$, and the operator

$$\sum_{j=0}^{r} \binom{r}{j} (-1)^j\, A^{r-j} X A^j\,, \tag{5.7}$$

clearly well defined on $\mathrm{Dom}(A^r)$, is bounded. We call the closure of (5.7) $\mathrm{ad}_A^r X$. The Leibniz rule can be generalized as usual.

Lemma 5.1. *Suppose that we are given $p, r \in \mathbb{N}$, a closed, densely defined operator A and $X, B_1, \dots, B_{p-1} \in \mathcal{B}(\mathcal{K})$ such that the operators $\operatorname{ad}_A^j X$ are bounded for all j, $1 \le j \le r$, and*

$$[\,A, B_1\,] = \cdots = [\,A, B_{p-1}\,] = 0.$$

Then $\operatorname{ad}_A^r(XB_1X\dots B_{p-1}X)$ is bounded and its norm is estimated from above by

$$\begin{aligned} \prod_{i=1}^{p-1} \|B_i\| \sum_{\substack{\nu\in\mathbb{Z}_+^r \\ \nu_1+2\nu_2+\cdots+r\nu_r=r}} & \frac{r!}{\prod_{j=1}^{r} (j!)^{\nu_j}\,\nu_j!}\, p(p-1)\dots(p-|\nu|+1) \\ & \times\, \|X\|^{p-|\nu|} \prod_{j=1}^{r} \|\operatorname{ad}_A^j X\|^{\nu_j}\,. \end{aligned} \tag{5.8}$$

Proof. Let us recall a formula of differentiation of functions,

$$\begin{aligned} \partial_x^r\, h(x)^p = \sum_{\substack{\nu\in\mathbb{Z}_+^r \\ \nu_1+2\nu_2+\cdots+r\nu_r=r}} & \frac{r!}{\prod_{j=1}^{r} (j!)^{\nu_j}\,\nu_j!}\, p(p-1)\dots(p-|\nu|+1) \\ & \times\, h(x)^{p-|\nu|} \prod_{j=1}^{r} \big(\partial_x^j\, h(x)\big)^{\nu_j}\,. \end{aligned} \tag{5.9}$$

In our case ad_A plays the role of differentiation. However, one cannot use the formula (5.9) directly since generally $\operatorname{ad}_A^i X$ and $\operatorname{ad}_A^j X$, for $i \neq j$, don't commute. Nevertheless we have, according to the generalized Leibniz rule,

$$\begin{aligned} &\operatorname{ad}_A^r(XB_1X\dots B_{p-1}X) = \\ &= \sum_{\mu\in\mathbb{Z}_+^p,\ |\mu|=r} \binom{r}{\mu} \left(\operatorname{ad}_A^{\mu_1} X\right) B_1 \left(\operatorname{ad}_A^{\mu_2} X\right)\dots B_{p-1} \left(\operatorname{ad}_A^{\mu_p} X\right). \end{aligned} \tag{5.10}$$

Estimating the norm of each summand in (5.10) by

$$\binom{r}{\mu} \prod_{i=1}^{p-1} \|B_i\| \prod_{j=1}^{r} \|\operatorname{ad}_A^j X\|^{\nu_j}$$

and grouping together the terms with the same powers $\mu_1, \dots, \mu_p$, up to a permutation, one arrives obviously at the same coefficients as in (5.9). $\square$

In the subsequent applications we substitute the time derivative for the operator A. Set $D := (-i/\omega)\,\partial_t \otimes 1$; this is to say, when identifying $\mathcal{K} \equiv l^2(\mathbb{Z}\times\mathbb{N})$,

$$Dh_n = n_1\, h_n, \quad \forall h = (h_n) \in \operatorname{Dom}(D) \subset \mathcal{K}. \tag{5.11}$$

It is clear that D is reducible by the projectors P and Q. If $V(t) \in C^r$ then the operator-valued function $V^{(j)}(\omega t)$, with $0 \le j \le r$, induces naturally the bounded operator $\operatorname{ad}_D^j V \in \mathcal{B}(\mathcal{K})$, and we have

$$\left(\operatorname{ad}_D^j V\right)_{mn} = (m_1 - n_1)^j\; V_{mn}\,.$$

This is a standard remark that the differentiability or, more generally, the boundedness of $\operatorname{ad}_D^r X$ induces a decay of matrix entries of an operator $X \in \mathcal{B}(\mathcal{K})$. More precisely, if X and $\operatorname{ad}_D^r X$ are bounded then

$$|X_{mn}| \le \max\{\|X\|,\ 2^r\,\|\operatorname{ad}_D^r X\|\}\ (1 + |m_1 - n_1|)^{-r}\,. \tag{5.12}$$

Particularly this applies to $V \in \mathcal{B}(\mathcal{K})$.

To proceed further we employ the diophantine estimate (3.11).

Lemma 5.2. *Suppose that, in the strong sense, $V(t) \in C^r$, and $r \geq 2$. Then for any p-tuple $(s_1, \ldots, s_p) \in \mathbb{N}^p$ and $q \in \mathbb{N}$ such that $q\sigma \leq r\alpha$ it holds true that*

(i) $\Gamma_0^q P_S(\hat{V}\Gamma_0^{s_1} P_R \hat{V} \ldots \Gamma_0^{s_p} P_R \hat{V}) P_S \Gamma_0^{-q} \in \mathcal{B}(\mathrm{Ran}(Q))$,

(ii) $\Gamma_0^q P_S(\hat{V}\Gamma_0^{s_1} P_R \hat{V} \ldots \Gamma_0^{s_p} P_R V) f$ *is well defined.*

Both in (i) and (ii) the value $p = 0$ is allowed and then the corresponding expressions read $\Gamma_0^q P_S \hat{V} P_S \Gamma_0^{-q}$ and $\Gamma_0^q P_S V f$, respectively.

Proof. First we establish the inequality

$$|(\hat{V}\Gamma_0^{s_1} P_R \hat{V} \ldots \Gamma_0^{s_p} P_R \hat{V})_{mn}| \leq \left(\frac{2}{\omega}\right)^{\sum s_j} C_V\, (1 + |m_1 - n_1|)^{-r} \tag{5.13}$$

where $C_V \equiv C_V(p, r)$ is a constant. Indeed, according to Lemma 5.1, $\mathrm{ad}_{\hat{D}}^{\,r}(\hat{V}\Gamma_0^{s_1} P_R \hat{V} \ldots \Gamma_0^{s_p} P_R \hat{V})$ is bounded for $[\,\hat{D}, \Gamma_0 P_R\,] = 0$ and $\mathrm{ad}_{\hat{D}}^{\,j}\,\hat{V}$ are bounded, $1 \leq j \leq r$. When applying the bound (5.8) observe that

$$\prod_{j=1}^{p} \|\Gamma_0^{s_j} P_R\| \leq \left(\frac{2}{\omega}\right)^{\sum s_j}.$$

Now it suffices to use (5.12).

We shall verify the item (i); the proof of (ii) is quite similar. Set temporarily

$$Y := \Gamma_0^q P_S \hat{V} \Gamma_0^{s_1} P_R \hat{V} \ldots \Gamma_0^{s_p} P_R \hat{V} P_S \Gamma_0^{-q}.$$

Suppose that $m, n \in \mathcal{S}$. By the inequality (5.13) we have

$$|Y_{mn}| \leq \mathrm{const}\, |F_m - F|^{-q}\, (1 + |m_1 - n_1|)^{-r}\, |F_n - F|^q. \tag{5.14}$$

The diagonal of Y is bounded and so it suffices to estimate only the off-diagonal part. Combining (5.14) with (3.11), (5.6) and (5.4) we get $(m \neq n)$

$$|Y_{mn}| \leq \mathrm{const} \left(\frac{\omega(1+\alpha)}{C_E}\right)^r \gamma^{-q} \left(\frac{\omega}{2}\right)^q m_2^{q\sigma - r\alpha}\, |m_2 - n_2|^{-r} \leq \mathrm{const}'\, |m_2 - n_2|^{-r}.$$

Since $r \geq 2$ we deduce that both

$$\sup_{m \in \mathcal{S}} \sum_{n \in \mathcal{S}} |Y_{mn}| \quad \text{and} \quad \sup_{n \in \mathcal{S}} \sum_{m \in \mathcal{S}} |Y_{mn}|$$

are finite and, in accordance with the Schur-Holmgren criterion, the norm $\|Y\|$ is estimated from above by the maximal of these two numbers. □

As a straightforward consequence we get

Lemma 5.3. *Suppose that $V(t) \in C^r$, $r \geq 2$. Then for any p-tuple $(s_1, \ldots, s_p) \in \mathbb{N}^p$ it holds true that*

$$\sum_{j=1}^{p} s_j \leq \frac{r\alpha}{\sigma} \Longrightarrow \Gamma_0^{s_1} \hat{V} \ldots \Gamma_0^{s_{p-1}} \hat{V} \Gamma_0^{s_p} Q V f \quad \textit{is well defined.}$$

Proof. Write, for each j,

$$\Gamma_0^{s_j} = \Gamma_0^{s_j} P_R + \Gamma_0^{s_j} P_S$$

and expand the resulting expression getting this way 2^p summands. Lemma 5.2 ad(i) can be used to move, in each summand, those powers $\Gamma_0^{s_{i_j}}$ which are accompanied by the projector P_S from the left to the right. Thus the problem reduces finally to the existence of the vector

$$\Gamma_0^q P_S \hat{V} \Gamma_0^{s_k} P_R \dots \Gamma_0^{s_{p-1}} P_R \hat{V} \Gamma_0^{s_p} P_R V f$$

where $1 \leq k \leq p+1$ (by definition, the expression reads $\Gamma_0^q P_S V f$ for $k = p+1$) and $q = \sum s_{i_j}$. By assumption, $q \leq r\alpha/\sigma$ and thus Lemma 5.2 ad(ii) proves the result. □

Combining Proposition 4.3 with Lemma 5.3 we get

Proposition 5.4. *Suppose that $V(t) \in C^r$, with $r \geq 2$, and $\ell \in \mathbb{N}$ obeys $\sigma\ell \leq r\alpha$. Then the Rayleigh-Schrödinger coefficients $\lambda_1, \dots, \lambda_\ell \in \mathbb{R}$ and $g_1, \dots, g_\ell \in$* Ran(Q) *exist and represent the unique solution to the system of equations (4.5) (or, equivalently, (4.6)).*

Remarks. (1) The existence of the RS coefficients is guaranteed by the differentiability of $V(t)$; the strong continuity is generally not sufficient. One can construct, for almost all $\omega > 0$, an operator-valued function $V(t)$ which is strongly continuous and such that already the coefficient λ_2 doesn't exist. This is the subject of Appendix B.

(2) For the choice of σ and τ specified in (3.9) it holds clearly true that $\sigma\ell < \tau(\ell+2)$ and hence the assumptions of Proposition 5.4 are fulfilled. So the first part of Theorem 2.1 has been proven. On the other hand, this comparison suggests that the assumption $r > 16/\alpha$ of Theorem 2.1 is very probably not optimal and could be improved.

6. Solution of the eigen-vector equation

In the sequel we adopt a standard simplification which doesn't imply any loss of generality. Namely, replacing V by $V - V_{\eta\eta}$ means just the shift of the spectrum,

$$\mathrm{Spec}(K + \beta(V - V_{\eta\eta})) = \mathrm{Spec}(K + \beta V) - \beta V_{\eta\eta}\,,$$

while all eigen-vectors stay untouched. Also the assumptions of Theorem 2.1 are not influenced by this replacement; particularly the coefficient λ_2 given in (2.5) suffers no change (as $Qf = 0$). So from now on we assume that

$$V_{\eta\eta} \equiv \langle Vf, f\rangle = 0 \iff Vf \in \mathrm{Ran}(Q)\,. \tag{6.1}$$

This implies also that the RS coefficients are expressed explicitly by the formulas (4.9) and (4.10). We rewrite the equalities (3.2) and (3.3) as

$$\lambda = \beta\,\langle Vf, g\rangle\,, \tag{6.2}$$

$$(\hat{K} + \beta\,\hat{V} - F - \lambda)g = -\beta\,Vf\,. \tag{6.3}$$

Our task in this section is to solve the equation (6.3), at least for particular values of β and λ. The first observation is that (6.3) can be reduced to the subspace $\mathrm{Ran}(P_S)$. We define

$$W(\beta,\lambda) := V(1+\beta\,\Gamma_\lambda P_R V)^{-1} = (1+\beta\,V\Gamma_\lambda P_R)^{-1}V\,. \tag{6.4}$$

Using (3.6) and (5.6) we get an estimate valid for $|\lambda| < \omega/2$,

$$\|\Gamma_\lambda P_R\| = \|(1-\lambda\,\Gamma_0 P_R)^{-1}\Gamma_0 P_R\| \leq \Big(\frac{\omega}{2} - |\lambda|\Big)^{-1}\,.$$

Hence $W(\beta,\lambda)$ is a well defined bounded operator and even analytically depending on (β,λ) in the domain

$$|\beta| \leq \frac{1}{12}\,\omega\|V\|^{-1}, \quad |\lambda| \leq \frac{1}{3}\,\omega\,, \tag{6.5}$$

and having the bound there

$$\|W(\beta,\lambda)\| \leq (1-|\beta|\|\Gamma_\lambda P_R\|\|V\|)^{-1}\|V\| \leq 2\|V\|\,. \tag{6.6}$$

To simplify the notation we set

$$W_S(\beta,\lambda) := P_S\,W(\beta,\lambda)\,P_S\,.$$

Lemma 6.1. *If* $g_S \in \mathrm{Ran}(P_S)\cap\mathrm{Dom}(\hat{K})$ *solves the equation*

$$\big(\hat{K} + \beta\,W_S(\beta,\lambda) - F - \lambda\big)g_S = -\beta\,P_S W(\beta,\lambda)f\,, \tag{6.7}$$

with β *and* λ *being restricted by (6.5), then*

$$g = \big(1 - \beta\,\Gamma_\lambda P_R W(\beta,\lambda)\big)g_S - \beta\,\Gamma_\lambda P_R W(\beta,\lambda)f$$

belongs to $\mathrm{Dom}(\hat{K})$ *and solves the equation (6.3).*

Proof. Obviously $g \in \mathrm{Dom}(\hat{K})$ since $\mathrm{Ran}(\Gamma_\lambda) = \mathrm{Dom}(\hat{K})$. Furthermore,

$$\begin{aligned}(\hat{K} + \beta\,\hat{V} - F - \lambda)\Gamma_\lambda P_R W(\beta,\lambda) &= (P_R + \beta\,\hat{V}\Gamma_\lambda P_R)W(\beta,\lambda)\\ &= QV - P_S W(\beta,\lambda)\,.\end{aligned}$$

Hence

$$\begin{aligned}(\hat{K} + \beta\,\hat{V} - F - \lambda)g &= (\hat{K} + \beta\,\hat{V} - F - \lambda)g_S - \beta(QV - P_S W(\beta,\lambda))g_S\\ &\quad - \beta(QV - P_S W(\beta,\lambda))f\\ &= (\hat{K} + \beta\,W_S(\beta,\lambda) - F - \lambda)g_S + \beta\,P_S W(\beta,\lambda)f - \beta\,Vf\\ &= -\,\beta\,Vf\,. \quad \square\end{aligned}$$

We are about to solve the reduced equation (6.7). Let us write, for the moment very formally,

$$(\hat{K} + \beta\,W_S(\beta,\lambda) - F - \lambda)^{-1} = (1 + \beta\,\Gamma(\beta,\lambda)\,W_S^{\mathrm{off}}(\beta,\lambda))^{-1}\Gamma(\beta,\lambda) \tag{6.8}$$

where

$$\Gamma(\beta,\lambda) := (\hat{K} + \beta\, W_S^{\mathrm{diag}}(\beta,\lambda) - F - \lambda)^{-1}\,.$$

Here we have used the obvious notation: $X^{\mathrm{off}} := X - X^{\mathrm{diag}}$ and X^{diag} is the diagonal part of an operator $X \in \mathcal{B}(\mathrm{Ran}(Q))$. The next step is to justify the equality (6.8) in which the diagonal and off-diagonal parts of $W_S(\beta,\lambda)$ have been separated. In order to treat the diagonal part we introduce another diophantine-like condition, this time in the parameters β and λ,

$$|F_n - F - \lambda + \beta\, W(\beta,\lambda)_{nn}| \geq \tilde{\psi}(n_2) \quad \text{for all } n \in \mathcal{S}, \tag{6.9}$$

with $\tilde{\psi}$ having been defined in (3.12). If $\tau \geq \sigma > 1$ then, in virtue of (3.11), the point $(\beta,\lambda) = (0,0)$ obeys the condition (6.9). Let us rewrite (6.9) in an operator form. For this sake we define, parallelly to the definition of D in (5.11), a self-adjoint unbounded operator L acting in $\mathcal{K} \equiv l^2(\mathbb{Z}\times\mathbb{N})$ by

$$Lh_n = n_2\, h_n, \quad \forall h = (h_n) \in \mathrm{Dom}(L) \subset \mathcal{K}\,. \tag{6.10}$$

The condition (6.9) is equivalent to

$$\|\Gamma(\beta,\lambda) L^{-\tau} P_S\| \leq \frac{2}{\gamma}\,. \tag{6.11}$$

Let us now focus on the off-diagonal part of $W_S(\beta,\lambda)$. First we prove an auxiliary estimate.

Lemma 6.2. *Suppose that A is a bounded, densely defined operator in $\mathcal{K}$, and $B, X \in \mathcal{B}(\mathcal{K})$ are such that $[\,A, B\,] = 0$, $\|B\|\|X\| < 1$, the operators $\mathrm{ad}_A^{\,j}\, X$ are bounded for $1 \leq j \leq p$, and*

$$\|B\| \max_{1\leq j\leq p} \|\,\mathrm{ad}_A^{\,j}\, X\| \leq 1\,.$$

Then

$$\|\,\mathrm{ad}_A^{\,p}\, X(1-BX)^{-1}\| \leq \frac{p!\,(2^{p+1}-1)}{(1-\|B\|\|X\|)^{p+1}} \max_{0\leq j\leq p} \|\,\mathrm{ad}_A^{\,j}\, X\|\,.$$

Proof. The case $p = 0$ is evident. Suppose that $p \geq 1$ and set temporarily

$$M := \max_{0\leq j\leq p} \|\,\mathrm{ad}_A^{\,j}\, X\|\,.$$

In virtue of Lemma 5.1 we have

$$\begin{aligned}
\|\,\mathrm{ad}_A^{\,p}\, X(1-BX)^{-1}\| &= \left\|\sum_{k=0}^{\infty} \mathrm{ad}_A^{\,p}\, X(BX)^k\right\| \\
&\leq \sum_{\substack{\nu\in\mathbb{Z}_+^p \\ \nu_1+2\nu_2+\cdots+p\nu_p=p}} \frac{p!}{\prod_{j=1}^{p} (j!)^{\nu_j}\, \nu_j!} \prod_{j=1}^{p} \|\,\mathrm{ad}_A^{\,j}\, X\|^{\nu_j} \\
&\qquad \times \sum_{k=0}^{\infty} (k+1)k\ldots(k+2-|\nu|)\, \|B\|^k \|X\|^{k+1-|\nu|} \\
&\leq \sum_{\substack{\nu\in\mathbb{Z}_+^p \\ \nu_1+2\nu_2+\cdots+p\nu_p=p}} \frac{p!}{\prod_{j=1}^{p} (j!)^{\nu_j}\, \nu_j!}\, M^{|\nu|}\, \frac{\|B\|^{|\nu|-1}|\nu|!}{(1-\|B\|\|X\|)^{|\nu|+1}} \\
&\leq \frac{p!\, M}{(1-\|B\|\|X\|)^{p+1}} \sum_{\substack{\nu\in\mathbb{Z}_+^p \\ \nu_1+2\nu_2+\cdots+p\nu_p=p}} \frac{|\nu|!}{\prod_{j=1}^{p} (j!)^{\nu_j}\, \nu_j!}\,.
\end{aligned}$$

Here we have used that, for $|x| < 1$ and $j \in \mathbb{Z}_+$,

$$\sum_{k=0}^{\infty} k(k-1)\dots(k-j+1)\,x^{k-j} = \frac{j!}{(1-x)^{j+1}}\,.$$

To finish the proof we estimate

$$\begin{aligned}\sum_{\substack{\nu\in\mathbb{Z}_+^p\\ \nu_1+2\nu_2+\dots+p\nu_p=p}} \frac{|\nu|!}{\prod_{j=1}^{p}(j!)^{\nu_j}\,\nu_j!} &< \sum_{k=0}^{p}\,\sum_{\nu\in\mathbb{Z}_+^p,\,|\nu|=k}\binom{k}{\nu}\prod_{j=1}^{p}\left(\frac{1}{j!}\right)^{\nu_j}\\ &= \sum_{k=0}^{p}\left(\sum_{j=1}^{p}\frac{1}{j!}\right)^{k}\\ &< 2^{p+1}-1\,. \qquad \square\end{aligned}$$

Lemma 6.2 applied to $W(\beta,\lambda)$ yields

$$\begin{aligned}\|\operatorname{ad}_D^r W(\beta,\lambda)\| &\le \frac{r!\,(2^{r+1}-1)}{(1-|\beta|\|\Gamma_\lambda P_R\|\|V\|)^{r+1}}\,\max_{0\le j\le r}\|\operatorname{ad}_D^j V\|\\ &\le r!\,2^{2r+2}\max_{0\le j\le r}\|\operatorname{ad}_D^j V\| \end{aligned}\tag{6.12}$$

where the couple (β,λ) obeys (6.5).

In accordance with (5.12), the existence of $\operatorname{ad}_D^r X$ implies a decay of the matrix entries of X. Below we derive some consequences of this fact. We consider also the situation when $X(z)$ is an analytic family of bounded operators.

Lemma 6.3. *Suppose that A is a closed, densely defined operator in $\mathcal{K}$, $\mathcal{U}\subset\mathbb{C}^N$ is open and $X(z)$, $z\in\mathcal{U}$, is an analytic family of bounded operators such that* $\operatorname{Ran}(X(z))\subset\operatorname{Dom}(A)$ *for all $z\in\mathcal{U}$. If the family $AX(z)$ is locally uniformly bounded on $\mathcal{U}$ then it is analytic.*

Proof. It is known (see VII§1.1 in [13]) that a family of bounded operators $Y(z)$ is analytic if and only if it is locally uniformly bounded and there exist two fundamental subsets $\mathcal{X}_1,\mathcal{X}_2\subset\mathcal{K}$ such that the functions $\langle h_2, Y(z)h_1\rangle$ are analytic for all $h_1\in\mathcal{X}_1$ and $h_2\in\mathcal{X}_2$. We apply this criterion to $Y(z)=AX(z)$, $\mathcal{X}_1=\mathcal{K}$ and $\mathcal{X}_2=\operatorname{Dom}(A^*)$. Then the functions

$$\langle h_2, AX(z)h_1\rangle = \langle A^*h_2, X(z)h_1\rangle$$

are manifestly analytic. $\square$

The symbol $\zeta(z)$ below stands for the Riemann zeta function,

$$\zeta(z) := \sum_{k=1}^{\infty} k^{-z}\,.$$

Lemma 6.4. *Suppose that $X \in \mathcal{B}(\mathcal{K})$, $\operatorname{ad}_D^r X$ is bounded for some $r \in \mathbb{N}$ and a number $\tau \in \mathbb{R}$ satisfies $\tau \le r\alpha$. It holds true that*

(i) $P_S X f \in \operatorname{Dom}(L^\tau)$ *and*

$$\|L^\tau P_S X f\| \le \sqrt{2\,\zeta(2r)} \left(\frac{\omega(1+\alpha)}{C_E}\right)^r \max\{\|X\|,\, 2^r \|\operatorname{ad}_D^r X\|\}\,, \tag{6.13}$$

(ii) *if $r \ge 2$ then* $\operatorname{Ran}(P_S X^{\mathrm{off}} P_S) \subset \operatorname{Dom}(L^\tau)$ *and*

$$\|L^\tau P_S X^{\mathrm{off}} P_S\| \le 2\,\zeta(r) \left(\frac{\omega(1+\alpha)}{C_E}\right)^r \max\{\|X\|,\, 2^r \|\operatorname{ad}_D^r X\|\}\,. \tag{6.14}$$

Suppose, in addition, that $X(z)$ is an analytic family on an open set $\mathcal{U} \subset \mathbb{C}^N$ and $\operatorname{ad}_D^r X(z)$ is locally uniformly bounded. Then, otherwise under the same assumptions, the families $L^\tau P_S X(z) f$ and $L^\tau P_S X^{\mathrm{off}}(z) P_S$ are analytic.

Proof. The inequalities (6.13) and (6.14) follow readily from (5.12) in combination with (5.5) or (5.4), respectively. For example, if $m, n \in \mathcal{S}$, $m \ne n$, then

$$|(L^\tau X)_{mn}| \le \left(\frac{\omega(1+\alpha)}{C_E}\right)^r \max\{\|X\|,\, 2^r \|\operatorname{ad}_D^r X\|\}\, m_2^{\tau - r\alpha}\, |m_2 - n_2|^{-r}\,.$$

Since $m_2^{\tau - r\alpha} \le 1$ and

$$\sum_{n \in \mathcal{S},\ n \ne m} |m_2 - n_2|^{-r} \le 2\,\zeta(r)\,,$$

the Schur-Holmgren criterion leads to (6.14). The verification of (6.13) is similar; instead of the Schur-Holmgren criterion one uses the equality

$$\|L^\tau P_S X f\|^2 = \sum_{n \in \mathcal{S}} |(L^\tau X)_{n\eta}|^2\,.$$

Concerning the second part of the lemma, the inequalities (6.13) and (6.14) imply respectively that the families $L^\tau P_S X(z) f$ and $L^\tau P_S X^{\mathrm{off}}(z) P_S$ are locally uniformly bounded on $\mathcal{U}$ and so, in virtue of Lemma 6.3, they are analytic. □

Now we can formulate an existence result.

Proposition 6.5. *Suppose that $V(t) \in C^r$, with $r \ge 2$, and a couple $(\beta, \lambda) \in \mathbb{R}^2$ obeys the diophantine estimate (6.9), i.e., $\|\Gamma(\beta,\lambda)\, L^{-\tau} P_S\| \le 2/\gamma$, with some τ, $0 \le \tau \le r\alpha$, and, in addition, it fulfills the inequalities*

$$|\beta| \le \min\left\{C_g(r)^{-1},\, \frac{1}{12}\,\omega\|V\|^{-1}\right\}, \quad |\lambda| \le \frac{\omega}{3}\,, \tag{6.15}$$

where

$$C_g(r) := \frac{1}{\gamma}\, 32\,\zeta(r)\, r!\, \left(\frac{8\omega(1+\alpha)}{C_E}\right)^r \max_{0 \le j \le r} \|\operatorname{ad}_D^j V\|\,.$$

Then the vector

$$g_S(\beta,\lambda) := -\beta\big(1 + \beta\,\Gamma(\beta,\lambda)\, W_S^{\mathrm{off}}(\beta,\lambda)\big)^{-1}\Gamma(\beta,\lambda)\, P_S W(\beta,\lambda) f \tag{6.16}$$

is well defined and the vector

$$g(\beta,\lambda) := \big(1-\beta\,\Gamma_\lambda P_R W(\beta,\lambda)\big)g_S(\beta,\lambda) - \beta\,\Gamma_\lambda P_R W(\beta,\lambda)f \tag{6.17}$$

solves the equation (6.3), i.e.,

$$(\hat{K}+\beta\,\hat{V}-F-\lambda)g(\beta,\lambda) = -\beta\,Vf\,. \tag{6.18}$$

Proof. Recall the estimates (6.6) and (6.12), and note that (6.11) implies

$$\mathrm{Dom}(L^\tau P_S) = \mathrm{Ran}(L^{-\tau}P_S) \subset \mathrm{Dom}(\Gamma(\beta,\lambda))\,.$$

According to Lemma 6.4 we have

$$P_S W(\beta,\lambda)f \in \mathrm{Dom}(\Gamma(\beta,\lambda)), \quad \mathrm{Ran}(W_S^{\mathrm{off}}(\beta,\lambda)) \subset \mathrm{Dom}(\Gamma(\beta,\lambda))\,,$$

and it holds

$$\begin{aligned}
\|\Gamma(\beta,\lambda)W_S^{\mathrm{off}}(\beta,\lambda)\| &\leq \|\Gamma(\beta,\lambda)\,L^{-\tau}P_S\|\,\|L^\tau W_S^{\mathrm{off}}(\beta,\lambda)\|\\
&\leq \frac{2}{\gamma}\cdot 2\zeta(r)\left(\frac{\omega(1+\alpha)}{C_E}\right)^r \max\{\|W(\beta,\lambda)\|,\, 2^r\|\,\mathrm{ad}_D^r\,W(\beta,\lambda)\|\}\\
&\leq \frac{1}{\gamma}\,16\zeta(r)\,r!\,\left(\frac{8\omega(1+\alpha)}{C_E}\right)^r \max_{0\leq j\leq r}\|\,\mathrm{ad}_D^j\,V\|\\
&= \frac{1}{2}\,C_g(r)\,.
\end{aligned}$$

This shows that $g_S(\beta,\lambda)$ is well defined.

Next we show that $g_S(\beta,\lambda)$ solves (6.7). It suffices to observe that $\mathrm{Ran}(\Gamma(\beta,\lambda)) \subset \mathrm{Dom}(\hat{K})$ and

$$\begin{aligned}
&(\hat{K}+\beta\,W_S(\beta,\lambda)-F-\lambda)\big(1+\beta\,\Gamma(\beta,\lambda)\,W_S^{\mathrm{off}}(\beta,\lambda)\big)^{-1}\Gamma(\beta,\lambda)P_S\\
&\quad = (\hat{K}-F-\lambda+\beta\,W_S^{\mathrm{diag}}(\beta,\lambda))\\
&\qquad \times\left(1-\beta\,\Gamma(\beta,\lambda)\,W_S^{\mathrm{off}}(\beta,\lambda)\big(1+\beta\,\Gamma(\beta,\lambda)\,W_S^{\mathrm{off}}(\beta,\lambda)\big)^{-1}\right)\Gamma(\beta,\lambda)P_S\\
&\qquad + \beta\,W_S^{\mathrm{off}}(\beta,\lambda)\big(1+\beta\,\Gamma(\beta,\lambda)\,W_S^{\mathrm{off}}(\beta,\lambda)\big)^{-1}\Gamma(\beta,\lambda)P_S\\
&\quad = P_S\,.
\end{aligned}$$

Hence

$$(\hat{K}+\beta\,W_S(\beta,\lambda)-F-\lambda)g_S(\beta,\lambda) = -\beta\,P_S\,W(\beta,\lambda)f\,.$$

The equality (6.18) is then a consequence of Lemma 6.1. ☐

7. More about the diophantine condition on β and λ

The diophantine condition (6.9) involves the diagonal of the operator $W(\beta,\lambda)$ whose definition (6.4) represents in fact the geometric series $V - \beta\, V\Gamma_\lambda P_R V + \dots$. We start by checking more closely the term $V\Gamma_\lambda P_R V$. Here is some additional notation. As one observes from (2.2), a matrix entry V_{mn} depends on m_1 and n_1 only through the difference $n_1 - m_1$; we write

$$V_{mn} =: V(n_1 - m_1, m_2, n_2).$$

Clearly,

$$\overline{V(k,p,q)} = V(-k,q,p).$$

Set, for $n \in \mathcal{S}$,

$$v_n(\lambda) := \sum_{k\in\mathbb{N}} \frac{|V(k,n_2,n_2)|^2}{\omega^2 k^2 - (F_n - F - \lambda)^2}\,. \tag{7.1}$$

In virtue of the condition (5.1), $v_n(\lambda)$ is well defined and even analytic for $|\lambda| \le \omega/3$, with the uniform bound

$$|v_n(\lambda)| \le \frac{\|V\|^2}{\omega^2}\left(\frac{1}{1-\left(\frac{5}{6}\right)^2} + \sum_{k\ge 2}\frac{1}{k^2-1}\right) = \frac{\|V\|^2}{\omega^2}\left(\frac{36}{11}+\frac{3}{4}\right).$$

It is also clear that on this domain all derivatives of $v_n(\lambda)$ are bounded uniformly and independently of $n \in \mathcal{S}$.

Lemma 7.1. *Suppose that $V(t) \in C^1$. Then there exists a constant $C_D > 0$ such that the inequality*

$$|(V\Gamma_\lambda P_R V)_{nn} + 2(F_n - F - \lambda)\, v_n(\lambda)| \le C_D\, n_2^{-\alpha} \tag{7.2}$$

holds true for all $n \in \mathcal{S}$ and all $\lambda \in \mathbb{R}$, $|\lambda| \le \omega/3$.

Proof. It suffices to verify (7.2) for the indices $n \in \mathcal{S}$ with sufficiently large components $n_2 \in \mathbb{N}$. So we assume that

$$1 \le c\, n_2^{\alpha} \quad \text{where} \quad c := C_E/3\omega(1+\alpha)\,. \tag{7.3}$$

Write temporarily $\mathcal{S}_\star := \mathcal{S} \cup \{\eta\}$. We express the diagonal element $(V\Gamma_\lambda P_R V)_{nn}$ as a sum,

$$(V\Gamma_\lambda P_R V)_{nn} = \sum_{m\notin\mathcal{S}_\star} |V_{nm}|^2\,(F_m - F - \lambda)^{-1}\,.$$

Observe that the partial sum, with the summation index satisfying $m \notin \mathcal{S}$ and $m_2 = n_2$, yields

$$\begin{aligned}
&\sum_{k\in\mathbb{Z},\ k\neq 0} |V(k,n_2,n_2)|^2\,(\omega k + F_n - F - \lambda)^{-1} \\
&\quad = \sum_{k\in\mathbb{N}} |V(k,n_2,n_2)|^2\left((\omega k + F_n - F - \lambda)^{-1} + (-\omega k + F_n - F - \lambda)^{-1}\right) \\
&\quad = -2\,(F_n - F - \lambda)\, v_n(\lambda)\,.
\end{aligned}$$

We split the rest (with the summation index $m \notin \mathcal{S}_\star$, $m_2 \neq n_2$) into two parts according to whether $|m_1 - n_1| \geq c\, n_2^\alpha$ or $|m_1 - n_1| < c\, n_2^\alpha$. In the first case we use the differentiability of $V(t)$, particularly the property

$$\sum_{m \in \mathbb{Z}\times\mathbb{N}} |m_1 - n_1|^2 |V_{nm}|^2 = \sum_{m \in \mathbb{Z}\times\mathbb{N}} |(\mathrm{ad}_D\, V)_{nm}|^2 \leq \|\,\mathrm{ad}_D\, V\|^2 \,,$$

and the fact that

$$|F_m - F - \lambda| \geq |F_m - F| - |\lambda| \geq \frac{\omega}{2} - \frac{\omega}{3} = \frac{\omega}{6}$$

holds true for $m \notin \mathcal{S}_\star$ and $|\lambda| \leq \omega/3$, to estimate

$$\begin{aligned}
\Bigg| \sum_{\substack{m \notin \mathcal{S}_\star \\ |m_1 - n_1| \geq c\, n_2^\alpha}} |V_{nm}|^2 \, (F_m - F - \lambda)^{-1} \Bigg| &\leq \frac{6}{\omega} \sum_{m \in \mathbb{Z}\times\mathbb{N}} \left(\frac{|m_1 - n_1|}{c\, n_2^\alpha} \right)^2 |V_{nm}|^2 \\
&\leq \frac{6}{\omega c^2} \,\|\,\mathrm{ad}_D\, V\|^2 \, n_2^{-2\alpha} \,.
\end{aligned}$$

In the second case we derive, using successively (5.1), (5.2) and (7.3),

$$\begin{aligned}
|F_m - F - \lambda| &\geq |F_m - F_n| - |F_n - F| - |\lambda| \\
&\geq |E_{m_2} - E_{n_2}| - \omega |m_1 - n_1| - \frac{\omega}{2} - \frac{\omega}{3} \\
&> \frac{C_E}{1+\alpha}\, n_2^\alpha - \omega c\, n_2^\alpha - \omega \\
&\geq \frac{C_E}{3(1+\alpha)}\, n_2^\alpha \,.
\end{aligned}$$

Hence

$$\begin{aligned}
\Bigg| \sum_{\substack{m \notin \mathcal{S}_\star \\ |m_1 - n_1| < c\, n_2^\alpha,\ m_2 \neq n_2}} |V_{nm}|^2 \, (F_m - F - \lambda)^{-1} \Bigg| &\leq \frac{3(1+\alpha)}{C_E}\, n_2^{-\alpha} \sum_m |V_{nm}|^2 \\
&\leq \frac{3(1+\alpha)}{C_E}\, \|V\|^2 \, n_2^{-\alpha} \,.
\end{aligned}$$

This completes the proof. $\square$

Let us now define, for $n \in \mathcal{S}$,

$$\tilde{w}_n(\beta, \lambda) := W(\beta, \lambda)_{nn} - 2\beta\, (F_n - F - \lambda)\, v_n(\lambda) \,, \tag{7.4}$$

$$w_n(\beta, \lambda) := \tilde{w}_n(\beta, \lambda)/(1 + 2\beta^2\, v_n(\lambda)) \,. \tag{7.5}$$

The diophantine estimate (6.9) can be rewritten as

$$|(F_n - F - \lambda)(1 + 2\beta^2\, v_n(\lambda)) + \beta\, \tilde{w}_n(\beta, \lambda)| \geq \tilde{\psi}(n_2) \quad \text{for all } n \in \mathcal{S}. \tag{7.6}$$

However, in the sequel we will replace (7.6) by a stronger condition, namely

$$|F_n - F - \lambda + \beta\, w_n(\beta, \lambda)| \geq \tilde{\psi}(n_2) \quad \text{for all } n \in \mathcal{S}. \tag{7.7}$$

Actually, (7.7) implies (7.6) since from the expression (7.1) one finds readily that $v_n(\lambda) > 0$ for all $n \in \mathcal{S}$ and all $\lambda \in \mathbb{R}$, $|\lambda| \leq \omega/3$.

Lemma 7.2. *The functions $w_n(\beta,\lambda)$, $n \in \mathcal{S}$, are analytic in a neighbourhood of the closed set*

$$|\beta| \le \frac{1}{12}\,\omega\|V\|^{-1}, \quad |\lambda| \le \frac{1}{3}\,\omega\,, \tag{7.8}$$

and on this set all their derivatives have bounds independent of $n \in \mathcal{S}$.

Suppose, moreover, that $V(t) \in C^1$. Then for each $\varepsilon > 0$ there exist $k_\star \in \mathbb{N}$ and $\delta_\star > 0$ such that

$$\sup_{n_2 \ge k_\star,\ |\beta| \le \delta_\star,\ |\lambda| \le \omega/3} |\partial_\beta w_n(\beta,\lambda)| < \varepsilon\,.$$

Proof. Concerning the uniform boundedness, one deduces from the formulas (7.4), (7.5) and from the properties of the functions $v_n(\lambda)$, as discussed above (see the definition (7.1)), that the problem reduces to an analogous assertion about the functions $W(\beta,\lambda)_{nn}$, $n \in \mathcal{S}$. But the latter case is quite obvious as the operator-valued function $W(\beta,\lambda)$ is analytic in the indicated domain (see the definition (6.4) and the related discussion).

Again from the definition (6.4) one finds that

$$W(\beta,\lambda) = V - \beta\, V\Gamma_\lambda P_R V + \beta^2\,(V\Gamma_\lambda P_R)^2 W(\beta,\lambda).$$

It follows readily that

$$w_n(\beta,\lambda) = V_{nn} - \beta\left((V\Gamma_\lambda P_R V)_{nn} + 2(F_n - F - \lambda)\,v_n(\lambda)\right) + \beta^2\,\rho_n(\beta,\lambda)\,, \tag{7.9}$$

where $\rho_n(\beta,\lambda)$, $n \in \mathcal{S}$, are analytic functions on the same domain and with all derivatives bounded there independently of n. Lemma 7.1 then implies the result. $\Box$

Denote by $\mathcal{D}$ the closed set determined by the countable family of diophantine inequalities,

$$\mathcal{D} := \{(\beta,\lambda) \in \mathbb{R}^2;\ (\beta,\lambda) \text{ satisfies (7.8) and (6.9)}\}. \tag{7.10}$$

In this definition the exponent τ (cf. (3.12)) can be, in principle, any real number but $\mathcal{D} \ne \emptyset$ is possible only for $\tau \ge 0$. Similarly, $\tilde{\mathcal{D}}$ is defined in the same manner but with the condition (6.9) (or, equivalently (7.6)) being replaced by the stronger condition (7.7). We know that if $\tau \ge \sigma > 1$ then $(0,0) \in \tilde{\mathcal{D}} \subset \mathcal{D}$. Next we are going to show that $\tilde{\mathcal{D}}$ contains, and so does $\mathcal{D}$, much more points than just the origin. But first we give two elementary lemmas.

Lemma 7.3. *Suppose that $h \in C^2(\mathbb{R})$ and $h''(x) \ge a > 0$ for all $x \in \mathbb{R}$. Then, for all $\varepsilon > 0$,*

$$|\{x \in \mathbb{R};\ |h(x)| < \varepsilon\}| \le 4\sqrt{\frac{\varepsilon}{a}}\,.$$

Proof. The function h has exactly one local extreme, namely a minimal value $h_{\min} = h(x_{\min})$, and, according to whether $h_{\min} \ge \varepsilon$ or $-\varepsilon < h_{\min} < \varepsilon$ or $h_{\min} \le -\varepsilon$, the set $h^{-1}(]-\varepsilon,\varepsilon[)$ is either empty or an open bounded interval or a union of two open bounded intervals. Even in the case when $h^{-1}(]-\varepsilon,\varepsilon[)$ is an open interval we split it by the extremal point $x_{\min}$ into two intervals. So it suffices to estimate the measure of an interval $[\,x_1,x_2\,]$ such that $h([\,x_1,x_2\,]) \subset [\,-\varepsilon,\varepsilon\,]$ and

h is monotone on $[\,x_1, x_2\,]$. For definiteness consider the case with h increasing. Then $h'(x_1) \geq 0$, $-\varepsilon \leq h(x_1) \leq h(x_2) \leq \varepsilon$, and we have

$$\begin{aligned} 2\varepsilon &\geq \int_{x_1}^{x_2} h'(s)\,ds \\ &= (x_2 - x_1)\,h'(x_1) + \int_{x_1}^{x_2} (x_2 - s)\,h''(s)\,ds \\ &\geq \frac{1}{2}\,(x_2 - x_1)^2\,a\,. \end{aligned}$$

Hence $|x_2 - x_1| \leq 2\,\sqrt{\varepsilon/a}$. $\square$

Lemma 7.4. *Suppose that $h \in C^2(\mathbb{R})$ and there are positive constants a, b, c such that*

$$|h(0)| \geq c, \quad |h'(0)| \leq b, \quad \textit{and} \quad |h''(x)| \geq a \quad \textit{for all } x \in \mathbb{R}.$$

Then for all $\varepsilon > 0$, $\varepsilon \leq \min\{b^2/a, c/2\}$, and all $\delta > 0$ it holds true that

$$|\{x \in [\,-\delta, \delta\,];\ |h(x)| < \varepsilon\}| \leq 8\delta\,\frac{b}{c}\,\sqrt{\frac{\varepsilon}{a}}\,.$$

Proof. Let us assume for definiteness that $h''(x) \geq a$ for all $x \in \mathbb{R}$. We distinguish two cases. First, assume that $h(0) \geq c$ (and $c \geq 2\varepsilon$). We apply Lemma 7.3 and the following observation. Consider the tangent line $y = h(0) + h'(0)\,x$ to the curve $y = h(x)$ and its intersection (x_0, ε) with the line $y = \varepsilon$. If $h^{-1}([\,-\varepsilon, \varepsilon\,]) \cap [\,-\delta, \delta\,] \neq \emptyset$ then, owing to the convexity,

$$\delta \geq |x_0| = |(h(0) - \varepsilon)/h'(0)| \geq (c - \varepsilon)/b \geq c/2b.$$

This way we get

$$|h^{-1}(]-\varepsilon, \varepsilon[) \cap [\,-\delta, \delta\,]| \leq 4\,\sqrt{\frac{\varepsilon}{a}} \leq 8\delta\,\frac{b}{c}\,\sqrt{\frac{\varepsilon}{a}}\,.$$

Second, assume that $h(0) \leq -c$. Then the set $h^{-1}(]-\varepsilon, \varepsilon[)$ is a union of two open bounded intervals. Consider, for example, that one on which h is increasing and denote it by $]x_1, x_2[$. If $[\,x_1, x_2\,] \cap [\,-\delta, \delta\,] \neq \emptyset$ then $0 < x_1 \leq \delta$ and, of course, $h(x_1) = -\varepsilon$, $h(x_2) = \varepsilon$. By convexity we have

$$\frac{2\varepsilon}{x_2 - x_1} = \frac{h(x_2) - h(x_1)}{x_2 - x_1} \geq \frac{h(x_1) - h(0)}{x_1} \geq \frac{c - \varepsilon}{\delta} \geq \frac{c}{2\delta}$$

and so $|x_2 - x_1| \leq 4\delta\varepsilon/c$. But the restriction $\varepsilon \leq b^2/a$ implies

$$|h^{-1}(]-\varepsilon, \varepsilon[) \cap [\,-\delta, \delta\,]| \leq 8\delta\,\frac{\varepsilon}{c} \leq 8\delta\,\frac{b}{c}\,\sqrt{\frac{\varepsilon}{a}}\,. \qquad \square$$

The following proposition gives a characterization of the set $\mathcal{D}$ which is determined, according to (7.10), by the diophantine-like condition (6.9).

Proposition 7.5. *Suppose that $V(t) \in C^1$ and the exponents τ and σ in (3.12) satisfy $\sigma > 1$ and $\tau > 2\sigma + 2$. Furthermore, suppose that $\varphi \in C^2(\mathbb{R})$, $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(0) \neq 0$. Set*

$$I(\varphi) := \left\{ \beta \in \mathbb{R};\ |\beta| \le \frac{1}{12}\,\omega\|V\|^{-1},\ |\varphi(\beta)| \le \frac{\omega}{3} \quad \textit{and} \quad (\beta, \varphi(\beta)) \in \mathcal{D} \right\}. \tag{7.11}$$

Then 0 is a point of density of the set $I(\varphi)$, i.e.,

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta}\, |I(\varphi) \cap [\,-\delta, \delta\,]| = 1\,.$$

Proof. Set (in this proof)

$$h_n(\beta) := F_n - F - \varphi(\beta) + \beta\, w_n(\beta, \varphi(\beta)), \quad n \in \mathcal{S}.$$

For $\delta > 0$ sufficiently small we have, as $\tilde{\mathcal{D}} \subset \mathcal{D}$,

$$\begin{aligned}
&[\,-\delta, \delta\,] \setminus I(\varphi) \subset \bigcup_{n \in \mathcal{S}} \Phi_n(\delta)\,, \quad \text{where} \\
&\Phi_n(\delta) := \{\beta \in [\,-\delta, \delta\,];\ |h_n(\beta)| < \tilde{\psi}(n_2)\}\,.
\end{aligned}$$

One finds that (cf. (7.9))

$$\begin{aligned}
&|h_n(0)| = |F_n - F| \ge \psi(n_2),\ h_n'(0) = w_n(0,0) = V_{nn}, \\
&h_n''(\beta) = -\varphi''(\beta) + 2\,\partial_\beta w_n(\beta, \varphi(\beta)) + 2\,\partial_\lambda w_n(\beta, \varphi(\beta))\,\varphi'(\beta) + O(|\beta|)\,.
\end{aligned}$$

From Lemma 7.2 and from the fact that $\varphi'(0) = 0$ we conclude that there exist $k_\star \in \mathbb{N}$ and $\delta_\star > 0$ such that

$$|h_n''(\beta)| \ge a, \quad \forall n \in \mathcal{S},\ n_2 \ge k_\star,\ \text{and}\ \forall \beta \in [\,-\delta_\star, \delta_\star\,],$$

where

$$a := |\varphi''(0)|/2\,. \tag{7.12}$$

Naturally we choose $\delta_\star > 0$ sufficiently small so that the inequalities $|\beta| \le \omega/(12\|V\|)$ and $|\varphi(\beta)| \le \omega/3$ are fulfilled for $|\beta| \le \delta_\star$.

Furthermore, since $\psi(k) > \tilde{\psi}(k)$, $\forall k \in \mathbb{N}$, there exists a sequence of positive numbers, $\{\beta_n\}_{n \in \mathcal{S}}$, such that $0 < \beta_n \le \delta_\star$ and

$$|h_n(\beta)| \ge \tilde{\psi}(n_2), \quad \forall \beta \in [\,-\beta_n, \beta_n\,],\ \forall n \in \mathcal{S}\,.$$

In other words, $\Phi_n(\delta) = \emptyset$ for $\delta \le \beta_n$. If necessary we increase the value $k_\star \in \mathbb{N}$ so that

$$\tilde{\psi}(k) \le \|V\|^2/a, \quad \forall k \ge k_\star\,. \tag{7.13}$$

Now we can apply Lemma 7.4, with $c = \psi(n_2)$, $b = \|V\|$, a given in (7.12) and $\varepsilon = \tilde{\psi}(n_2)$, to the set $\Phi_n(\delta)$. If $n_2 \ge k_\star$ then the assumption $\varepsilon \le \min\{b^2/a,\, c/2\}$ is satisfied owing to (7.13) and to the fact that $\psi(k) \ge 2\,\tilde{\psi}(k)$, $\forall k \in \mathbb{N}$. Hence

$$|\Phi_n(\delta)| \le 8\delta\, \frac{\|V\|}{\sqrt{a}}\, \frac{\sqrt{\tilde{\psi}(n_2)}}{\psi(n_2)} \le \text{const}\ 2\delta\, n_2^{-\frac{1}{2}\tau + \sigma}\,.$$

Summing up, provided

$$0 < \delta \leq \min_{n \in \mathcal{S},\ n_2 < k_\star} \beta_n \quad \text{and} \quad \delta \leq \delta_\star$$

(which implies that $\Phi_n(\delta) = \emptyset$ for $n_2 < k_\star$) we have the estimate

$$\begin{aligned} \frac{1}{2\delta}\, |[-\delta,\delta] \setminus I(\varphi)| &\leq \frac{1}{2\delta} \Bigg| \bigcup_{n \in \mathcal{S},\ n_2 \geq k_\star} \Phi_n(\delta) \Bigg| \\ &\leq \text{const} \sum_{n \in \mathcal{S},\ \beta_n < \delta} n_2^{-\frac{1}{2}\tau + \sigma} . \end{aligned}$$

Recall that the projection $\mathcal{S} \to \mathbb{N}\backslash\{\eta_2\}$ is one-to-one. Hence the sum $\sum_{n \in \mathcal{S}} n_2^{-\frac{1}{2}\tau+\sigma}$ converges. Since

$$\bigcap_{\delta > 0} \{n \in \mathcal{S};\ \beta_n < \delta\} = \emptyset$$

we get

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta}\, |[-\delta,\delta] \setminus I(\varphi)| = 0 . \qquad \square$$

8. Implicit equation, completion of the proof

Let us return to Proposition 6.5. Suppose that $V(t) \in C^r$, with $r \geq 2$, and that $\tau \leq r\alpha$, and denote by $\mathcal{D}(r)$ the intersection of the set $\mathcal{D}$ defined in (7.10) with the closed unit ball in $\mathbb{R}^2$ and with the closed set determined by the inequalities (6.15). In fact, $\mathcal{D}(r)$, as well as $\mathcal{D}$, depends also on the exponent τ, $\tau \geq 0$, (cf. (3.12)). Then for all $(\beta,\lambda) \in \mathcal{D}(r)$ the vector $g(\beta,\lambda)$ defined in (6.16) and (6.17) solves the equation (6.3). Recall that $\hat{V} = QVQ$; consequently

$$Vh = \hat{V}h + \langle Vf, h\rangle\, f, \quad \forall h \in \operatorname{Ran}(Q) .$$

Altogether this means that

$$(K + \beta V) g(\beta,\lambda) = (F + \lambda) g(\beta,\lambda) + \beta \langle Vf, g(\beta,\lambda)\rangle\, f - \beta V f .$$

Since $Kf = Ff$ we arrive at the equality

$$(K + \beta V)(f + g(\beta,\lambda)) = (F + \lambda)(f + g(\beta,\lambda)) + (G(\beta,\lambda) - \lambda) f \tag{8.1}$$

where

$$G(\beta,\lambda) := \beta\, \langle Vf, g(\beta,\lambda)\rangle . \tag{8.2}$$

Thus our final task, in order to get an eigen-value and an eigen-vector, is to solve the implicit equation

$$\lambda - G(\beta,\lambda) = 0 , \tag{8.3}$$

which is nothing but the equation (6.2).

We will solve (8.3) in a Lipschitz class. The notion of Lipschitz functions as well as their properties needed for our purposes are recalled in Appendix C. This also concerns the celebrated Whitney extension theorem [23]. We remind the reader that the target space is generally allowed to be a Banach space or, more particularly, a Banach algebra. When indicating that a function belongs to a Lipschitz class supported on a closed set we always assume tacitly that this concerns the corresponding restriction. We have to decide about the Lipschitz property of the vector-valued function $g(\beta,\lambda)$ defined on $\mathcal{D}(r)$. Looking at the formulas (6.16) and (6.17) one finds immediately that $\Gamma(\beta,\lambda)$ is the only operator-valued function occurring in the expressions which is not analytic (and so automatically Lipschitz).

Lemma 8.1. *For all $\ell \in \mathbb{Z}_+$, the function $\Gamma(\beta,\lambda)\, L^{-\tau(\ell+2)} P_S$ belongs to the Lipschitz class $\mathrm{Lip}(\ell+1, \mathcal{D} \cap \bar{B}_1)$ where $\bar{B}_1 \subset \mathbb{R}^2$ is the closed unit ball.*

Proof. Set temporarily

$$\phi_n(\beta,\lambda) := F_n - F - \lambda + \beta\, W(\beta,\lambda)_{nn}, \quad n \in \mathcal{S};$$

hence $\Gamma(\beta,\lambda)_{mn} = \phi_m(\beta,\lambda)^{-1}\, \delta_{mn}$. Owing to (5.6), the operator-valued function $(\hat{K} - F - \lambda + \beta\, W^{\mathrm{diag}}(\beta,\lambda))P_S$ is bounded and analytic on a neighbourhood of the closed set determined by (7.8), and so it belongs to $\mathrm{Lip}(\ell+1, \mathcal{D} \cap \bar{B}_1)$; denote by M_ℓ its Lipschitz norm. This implies that ($M(\cdot)$ stands for the Lipschitz norm)

$$\phi_n \in \mathrm{Lip}(\ell+1, \mathcal{D} \cap \bar{B}_1) \quad \text{and} \quad M(\phi_n) \le M_\ell \quad \text{for all } n \in \mathcal{S}\,.$$

Since $|\phi_n(\beta,\lambda)| \ge (\gamma/2)\, n_2^{-\tau}$ (cf. (6.9)) one can apply Proposition C.5, with the constant $C_L(2,\ell)$ redenoted as $C(\ell)$, to conclude that

$$M(\phi_n(\beta,\lambda)^{-1}) \le C(\ell)\, M_\ell^{\ell+1} \left(\frac{2}{\gamma}\, n_2^{\tau}\right)^{\ell+2}, \ \forall n \in \mathcal{S}.$$

This completes the proof for

$$M(\Gamma(\beta,\lambda)\, L^{-\tau(\ell+2)} P_S) \le C(\ell)\, M_\ell^{\ell+1} \left(\frac{2}{\gamma}\right)^{\ell+2} < \infty\,. \qquad \square$$

Lemma 8.2. *Suppose that $V(t) \in C^r$, with $r \ge 2$ and $0 \le \tau(\ell+2) \le r\alpha$, and $\ell \in \mathbb{Z}_+$. Then the vector-valued function $g(\beta,\lambda)$ defined in (6.17) belongs to the class $\mathrm{Lip}(\ell+1, \mathcal{D}(r))$.*

Proof. The function $\Gamma_\lambda P_R W(\beta,\lambda)$ is analytic in a neighbourhood of $\mathcal{D}(r)$ and so it belongs to the Lipschitz class of any order. Hence, in virtue of the relation (6.17) and Proposition C.4, it suffices to verify the assertion for the function $g_S(\beta,\lambda)$ instead of $g(\beta,\lambda)$. Here the Banach algebra in question is $\mathcal{B}(\mathcal{K})$. The fact that the expressions involve also $\mathcal{K}$-valued functions does not mean a serious complication: either one can modify, in an obvious way, Proposition C.4 or one can replace everywhere vectors $h \in \mathcal{K}$ by the rank-one operators $\tilde{h} \in \mathcal{B}(\mathcal{K})$, $\tilde{h}x := \langle f, x\rangle h$ (e.g., f would be replaced by P). Furthermore, from Lemma 6.4 we deduce that the functions $L^{\tau(\ell+2)}\, W_S^{\mathrm{off}}(\beta,\lambda)$ and $L^{\tau(\ell+2)}\, P_S W(\beta,\lambda) f$ are analytic as well. Checking the formula (6.16) one concludes readily from Lemma 8.1, Proposition C.4 and Proposition C.5 that $g_S(\beta,\lambda)$ belongs indeed to the indicated Lipschitz class. $\square$

Let us add a remark to Lemma 8.2. From the proof and from the formulas (6.16), (6.17) it is quite obvious that the functions $\beta^{-1}\, g_S(\beta,\lambda)$ and $\beta^{-1}\, g(\beta,\lambda)$ belong to $\mathrm{Lip}(\ell+1, \mathcal{D}(r))$. If $\tau \ge \sigma$ then $(0,0) \in \mathcal{D}(r)$ and we have

$$\begin{aligned} &\beta^{-1}\, g_S(\beta,\lambda)\big|_{(\beta,\lambda)=(0,0)} = -\Gamma_0 P_S V f\,, \\ &\beta^{-1}\, g(\beta,\lambda)\big|_{(\beta,\lambda)=(0,0)} = -\Gamma_0 P_S V f - \Gamma_0 P_R V f = -\Gamma_0 V f\,. \end{aligned}$$

The set $\mathcal{D}(r)$ is closed and so we can apply the Whitney extension theorem to the function $\beta^{-1}\, g(\beta,\lambda)$. As a consequence we get an extension $\tilde{g}(\beta,\lambda) \in \mathrm{Lip}(\ell+1, \mathbb{R}^2)$

of the function $g(\beta,\lambda)$ itself. Then, according to the formula (8.2), the function $G(\beta,\lambda)\in \mathrm{Lip}(\ell+1,\mathcal{D}(r))$ as well and

$$\tilde{G}(\beta,\lambda) := \beta\,\langle Vf, \tilde{g}(\beta,\lambda)\rangle \in \mathrm{Lip}(\ell+1,\mathbb{R}^2)\subset C^{\ell}(\mathbb{R}^2)$$

is an extension of it. Moreover, the previous remark implies that the function $\beta^{-2}\,\tilde{G}(\beta,\lambda)$ belongs to the class $\mathrm{Lip}(\ell+1,\mathbb{R}^2)$, too. Consequently, (if $\tau\geq\sigma$)

$$\partial_\lambda^j\tilde{G}(0,0)=0 \quad\text{and}\quad \partial_\beta\partial_\lambda^k\tilde{G}(0,0)=0 \quad \text{for } j,k\in\mathbb{Z}_+,\ j\leq\ell \text{ and } k\leq\ell-1, \tag{8.4}$$

and, if $\ell\geq 2$,

$$\partial_\beta^2\tilde{G}(0,0) = 2\beta^{-2}\,\tilde{G}(\beta,\lambda)\Big|_{(\beta,\lambda)=(0,0)} = -2\,\langle Vf,\Gamma_0 Vf\rangle\,.$$

Suppose that $\ell\geq 1$. Instead of (8.3) we shall consider the implicit equation in $\mathbb{R}^2$, this is to say with the extended function $\tilde{G}\in C^{\ell}(\mathbb{R}^2)$,

$$\lambda-\tilde{G}(\beta,\lambda)=0. \tag{8.5}$$

Since

$$\lambda-\tilde{G}(\beta,\lambda)\Big|_{(\beta,\lambda)=(0,0)}=0,\quad \partial_\lambda(\lambda-\tilde{G}(\beta,\lambda))\Big|_{(\beta,\lambda)=(0,0)}=1,$$

the implicit mapping theorem guarantees the existence of $\beta_\star>0$ and of a unique function $\tilde{\lambda}\in C^{\ell}([\,-\beta_\star,\beta_\star\,])$ such that

$$\tilde{\lambda}(0)=0 \quad\text{and}\quad \tilde{\lambda}(\beta)=\tilde{G}(\beta,\tilde{\lambda}(\beta)) \quad \text{for all } \beta\in[\,-\beta_\star,\beta_\star\,]\,. \tag{8.6}$$

Let us calculate the lowest order derivatives of $\tilde{\lambda}$:

$$\tilde{\lambda}'(\beta)=\big(1-\partial_\lambda\tilde{G}(\beta,\tilde{\lambda}(\beta))\big)^{-1}\partial_\beta\tilde{G}(\beta,\tilde{\lambda}(\beta))\,, \tag{8.7}$$

and so $\tilde{\lambda}'(0)=0$. If $\ell\geq 2$ then

$$\tilde{\lambda}''(0)=\partial_\beta^2\tilde{G}(0,0)=-2\,\langle Vf,\Gamma_0 Vf\rangle\,. \tag{8.8}$$

Proposition 8.3. *Suppose that $V(t)\in C^r$, with $r\geq 2$ and $\tau(\ell+2)\leq r\alpha$, and $\tau\geq\sigma>1$, $\ell\in\mathbb{N}$. Then there exist $\beta_\star>0$ and a solution $\tilde{\lambda}\in\mathrm{Lip}(\ell+1,[\,-\beta_\star,\beta_\star\,])$ of the implicit equation (8.5), i.e., the equalities (8.6) hold. Furthermore, the* $\mathrm{Ran}(Q)$*-valued function $\tilde{g}(\beta,\tilde{\lambda}(\beta))$, too, belongs to the class* $\mathrm{Lip}(\ell+1,[\,-\beta_\star,\beta_\star\,])$.

If, for some $\beta\in[\,-\beta_\star,\beta_\star\,]$, $(\beta,\tilde{\lambda}(\beta))\in\mathcal{D}$ then $F+\tilde{\lambda}(\beta)$ is an eigen-value of $K+\beta V$ corresponding to the eigen-vector $f+g(\beta,\tilde{\lambda}(\beta))$.

Proof. We already know that $\tilde{\lambda}\in C^{\ell}([\,-\beta_\star,\beta_\star\,])$. To complete the proof we have to show that $\tilde{\lambda}$ even belongs to $\mathrm{Lip}(\ell+1,[\,-\beta_\star,\beta_\star\,])$ or, equivalently, $\tilde{\lambda}^{(\ell)}\in\mathrm{Lip}(1,[\,-\beta_\star,\beta_\star\,])$. Let us first specify more precisely the choice of $\beta_\star>0$. We can assume, because of (8.4), that

$$|\partial_\lambda\tilde{G}(\beta,\tilde{\lambda}(\beta))|\leq\frac{1}{2},\quad \forall\beta\in[\,-\beta_\star,\beta_\star\,]\,.$$

Furthermore, since $\tilde{\lambda}(0)=0$, we require the points $(\beta,\tilde{\lambda}(\beta))$, with $\beta\in[-\beta_\star,\beta_\star]$, to satisfy the inequalities (6.15) and, at the same time, to belong to the unit ball $\bar{B}_1$. In other words, if $\beta\in[-\beta_\star,\beta_\star]$ and $(\beta,\tilde{\lambda}(\beta))\in\mathcal{D}$ then $(\beta,\tilde{\lambda}(\beta))\in\mathcal{D}(r)$.

In virtue of (8.7) we have

$$\sum_{j=1}^{\ell}\binom{\ell-1}{j-1}\tilde{\lambda}^{(j)}(\beta)\,\frac{d^{\ell-j}}{d\beta^{\ell-j}}\big(1-\partial_\lambda\tilde{G}(\beta,\tilde{\lambda}(\beta))\big)=\frac{d^{\ell-1}}{d\beta^{\ell-1}}\,\partial_\beta\tilde{G}(\beta,\tilde{\lambda}(\beta))\,. \tag{8.9}$$

Deduce from Proposition C.6 and from the fact that $\tilde{\lambda}\in\mathrm{Lip}(\ell,[-\beta_\star,\beta_\star])$ that

$$\partial_\beta^j\partial_\lambda^k\tilde{G}(\beta,\tilde{\lambda}(\beta))\in\mathrm{Lip}(\ell-j-k+1,[-\beta_\star,\beta_\star])\quad\text{if } 1\le j+k\le\ell.$$

One can express $\tilde{\lambda}^{(\ell)}$ from the identity (8.9); according to our choice of $\beta_\star$, $|1-\partial_\lambda\tilde{G}(\beta,\tilde{\lambda}(\beta))|\ge 1/2$. Now the usual rules of differentiation jointly with Proposition C.5 and Proposition C.4 imply that $\tilde{\lambda}^{(\ell)}\in\mathrm{Lip}(1,[-\beta_\star,\beta_\star])$.

This is also because of Proposition C.6 that we can claim that the composed function $\tilde{g}(\beta,\tilde{\lambda}(\beta))$ belongs to $\mathrm{Lip}(\ell+1,[-\beta_\star,\beta_\star])$. The final part of the assertion can be seen immediately from the equality (8.1) for it holds, by our choice of $\beta_\star$ specified above: if $\beta\in[-\beta_\star,\beta_\star]$ and $(\beta,\tilde{\lambda}(\beta))\in\mathcal{D}$ then

$$\tilde{\lambda}(\beta)=\tilde{G}(\beta,\tilde{\lambda}(\beta))=G(\beta,\tilde{\lambda}(\beta))\,.\qquad\square$$

Proof of Theorem 2.1. The first part of the theorem has been already proven in Section 5 – see Remark (2) at the end of the section. All the steps needed to show the second part, too, have been already stated and so we have just to summarize them. We make the choice of σ and τ as specified in (3.9). Proposition 6.5 guarantees the existence of a solution $g(\beta,\lambda)$ of the eigen-vector equation (6.3) provided (β,λ) belongs to $\mathcal{D}(r)$, a closed set introduced in the beginning of this section. Consider now the function $\tilde{\lambda}\in\mathrm{Lip}(\ell+1,[-\beta_\star,\beta_\star])$, as described in Proposition 8.3. Set

$$I:=[-\beta_\star,\beta_\star]\cap I(\tilde{\lambda})\,,$$

with $I(\varphi)$ having been defined in (7.11). Denote by $F(\beta)$ the restriction of the function $F+\tilde{\lambda}(\beta)$ to the set I and by $f(\beta)$ the restriction of $f+\tilde{g}(\beta,\tilde{\lambda}(\beta))$ to the same set. According to Proposition 8.3,

$$(K+\beta V)f(\beta)=F(\beta)f(\beta)\quad\text{for all }\beta\in I\,.$$

Since ℓ, as specified in Theorem 2.1, fulfills $\ell\ge 2$, and since $\tilde{\lambda}(0)=\tilde{\lambda}'(0)=0$, $\tilde{\lambda}''(0)=2\lambda_2\ne 0$ (cf. (8.7) and (8.8)), Proposition 7.5 tells us that 0 is a point of density of I. Finally, we know, again from Proposition 8.3, that both $F(\beta)$ and $f(\beta)$ belong to the Lipschitz class $\mathrm{Lip}(\ell+1,I)$. According to Lemma 4.1, the same is true for $(K+\beta V)f(\beta)$. Moreover, since $g(\beta,\lambda)\in\mathrm{Ran}(Q)$ we have $\langle f,f(\beta)\rangle=1$ for all $\beta\in I$. Then, as explained in Section 4, the coefficients from the asymptotic expansion of the functions $F(\beta)$ and $f(\beta)$ at $\beta=0$ obey the equations (4.5) (or, equivalently (4.6)). To complete the proof we note that Proposition 5.4 ensures the existence and uniqueness of the solution to this system of equations and Proposition 4.3 gives its explicit form coinciding with the standard formulas known for RS series.

Appendix A. Density of the spectrum for almost all frequencies

Proposition A.1. *Suppose that a set* $\mathcal{E} \subset \mathbb{R}$ *fulfills* $\sup \mathcal{E} = +\infty$. *Then the set* $\omega\mathbb{Z} + \mathcal{E}$ *is dense in* $\mathbb{R}$ *for almost all* $\omega \in \mathbb{R}$ *(in the Lebesgue sense).*

As $-\mathbb{Z} = \mathbb{Z}$ we can consider only positive values of ω. Furthermore, we make use of the facts that the positive half-line can be covered by a countable union of open bounded intervals and that the countable system of open intervals with rational endpoints forms a basis of the topology in $\mathbb{R}$. We conclude from this that the following proposition, seemingly weaker, is in fact equivalent to Proposition A.1.

Proposition A.2. *Suppose that we are given an open interval* $]a, b[$, $0 < a < b < \infty$, *and a compact interval* $[\,u, v\,]$. *Then, under the same assumptions about the set* $\mathcal{E}$ *as in Proposition A.1, it holds*

$$(\omega\mathbb{Z} + \mathcal{E}) \cap [\,u, v\,] \neq \emptyset \quad \textit{for almost all } \omega \in]a, b[\,.$$

Lemma A.3. *Suppose that* $\mathcal{E}$ *is the same as in Proposition A.1,* $[\,u, v\,]$ *is a compact interval,* $\mathcal{U} \subset]v - u, +\infty[$ *is an open set and* $|\mathcal{U}| < \infty$. *Then there exists* $x_\star \in \mathbb{R}$ *such that for all* $x > x_\star$ *one can find a closed set* $\mathcal{M}(x) \subset \mathcal{U}$ *with the properties:*

(1) $(x + \omega\mathbb{Z}) \cap [\,u, v\,] \neq \emptyset \quad$ *for all* $\omega \in \mathcal{M}(x)$,

(2) $|\mathcal{M}(x)| \geq \dfrac{1}{4}\,(v - u) \displaystyle\int_{\mathcal{U}} \frac{1}{s}\, ds\,.$

Proof. $\mathcal{U}$, as an open set, is at most countable disjoint union of open intervals. Since

$$\int_{\mathcal{U}} \frac{1}{s}\, ds \leq \frac{1}{v - u}\, |\mathcal{U}| < \infty$$

there exists a finite subunion $\mathcal{U}' = \bigcup \mathcal{U}_i \subset \mathcal{U}$, formed necessarily by bounded intervals, such that

$$\int_{\mathcal{U}'} \frac{1}{s}\, ds = \sum_i \int_{\mathcal{U}_i} \frac{1}{s}\, ds \geq \frac{1}{2} \int_{\mathcal{U}} \frac{1}{s}\, ds\,.$$

We will seek a family of closed subsets $\mathcal{M}_i(x) \subset \mathcal{U}_i$ so that, for each i, the properties (1) and (2) are valid for $\mathcal{M}_i(x)$ and $\mathcal{U}_i$ in the place of $\mathcal{M}(x)$ and $\mathcal{U}$, respectively, with the only difference: we replace the factor $1/4$ in (2) by $1/2$. Suppose that we are successful. Then the disjoint union $\mathcal{M}(x) := \bigcup_i \mathcal{M}_i(x)$ has all the required properties.

Fix an index i and write $\mathcal{U}_i =]a, b[$ where $0 < v - u \leq a < b < \infty$. Assume that

$$x > \max\left\{0, v, \frac{vb - ua}{b - a}\right\}.$$

Then

$$0 < \frac{x - u}{b} < \frac{x - v}{a} \leq \frac{x - v}{v - u}$$

and the union

$$\mathcal{M}_i(x) := \bigcup_{\substack{k \in \mathbb{N} \\ (x-u)/b < k < (x-v)/a}} \left[\frac{x - v}{k}, \frac{x - u}{k}\right]$$

is disjoint. Consequently,

$$\begin{aligned}|\mathcal{M}_i(x)| &= \sum_{(x-u)/b<k<(x-v)/a} \frac{v-u}{k} \\ &= (v-u)\,\log\frac{b}{a} + O(x^{-1}) \\ &\geq \frac{1}{2}\,(v-u)\,\log\frac{b}{a}\end{aligned}$$

for sufficiently large x. Moreover, if $\omega \in \mathcal{M}_i(x)$ then there exists $k \in \mathbb{N}$ such that $x - \omega k \in [\,u, v\,]$. $\square$

Proof of Proposition A.2. Clearly, $(x+\omega\mathbb{Z})\cap[\,u,v\,] \neq \emptyset$ for all ω, $0<\omega\leq v-u$, and all $x \in \mathbb{R}$. Consequently we can assume, without loss of generality, that $v-u \leq a$. Using Lemma A.3 we construct successively a sequence $\mathcal{M}(x_1), \mathcal{M}(x_2), \ldots$ formed by disjoint closed subsets of the interval $]a,b[$, with the points $x_k \in \mathcal{E}$, so that $\mathcal{M}(x_k)$ is related to the open set $\mathcal{U}_k = \,]a,b[\,\setminus\mathcal{N}_k$ where

$$\mathcal{N}_1 := \emptyset \quad \text{and} \quad \mathcal{N}_k := \mathcal{M}(x_1)\cup\cdots\cup\mathcal{M}(x_{k-1}) \quad \text{for } k \geq 2\,.$$

Set

$$\mathcal{N} := \bigcup_{k\in\mathbb{N}} \mathcal{N}_k = \bigcup_{k\in\mathbb{N}} \mathcal{M}(x_k)\,.$$

The property (1) implies

$$(\omega\mathbb{Z}+\mathcal{E})\cap[\,u,v\,] \neq \emptyset, \quad \forall\omega\in\mathcal{N}\,.$$

Furthermore, $|\mathcal{N}| = \lim|\mathcal{N}_k|$ and, owing to the property (2), we have

$$|\mathcal{N}_{k+1}| - |\mathcal{N}_k| = |\mathcal{M}_k| \geq \frac{1}{4}\int_{]a,b[\,\setminus\mathcal{N}_k} \frac{1}{s}\,ds\,. \tag{A.1}$$

Passing in (A.1) to the limit $k\to\infty$ we get

$$0 \geq \int_{]a,b[\,\setminus\mathcal{N}} \frac{1}{s}\,ds$$

and so $|]a,b[\,\setminus\mathcal{N}| = 0$. $\square$

Appendix B. A perturbation without Rayleigh-Schrödinger series

In this appendix we exhibit an example of a perturbation for which λ_2 given in (2.5) does not exist. The symbols H, K, E_k, F_n, e_k, f_n and V retain their meaning from Section 2. However we don't require anymore that the eigen-values E_k of the Hamiltonian H obey the gap condition (2.1). Instead we impose another restriction which has this time a multiplicative form. More precisely, we assume that there exist constants $C_M > 0$ and $\mu > 0$ such that

$$j<k \Longrightarrow \frac{E_k}{E_j} \geq C_M\left(\frac{k}{j}\right)^{\mu}. \tag{B.1}$$

Generally speaking, the conditions (2.1) and (B.1) are independent. However in some cases, for example when the eigen-values E_k grow polynomially, $E_k = \mathrm{const}\,k^{\mu}$, the condition (B.1) appears to be milder than (2.1). Actually, the condition (2.1) is satisfied provided $\mu > 1$ while (B.1) holds obviously for any μ positive.

Proposition B.1. *Suppose that the spectrum* $\mathrm{Spec}(H)$ *satisfies the condition (B.1). Then, for almost all* $\omega > 0$*, there exists a bounded self-adjoint perturbation* $V(t)$ *which is a* 2π*-periodic and strongly continuous function of* t *and such that the Rayleigh-Schrödinger coefficient* λ_2 *given in (2.5) doesn't exist, i.e., the series*

$$\sum_{n\in\mathbb{Z}\times\mathbb{N},\ n\neq\eta} \frac{1}{F_n - F_\eta}\,|V_{n\eta}|^2 \tag{B.2}$$

diverges, and this holds true for all $\eta \in \mathbb{Z}\times\mathbb{N}$*.*

Let us introduce yet another condition. Namely, one requires that there exist constants $C_M > 0$ and $\mu > 0$ such that

$$j < k \Longrightarrow \frac{E_k}{E_j} \geq C_M\, \exp\bigl(\mu(k-j)\bigr)\,. \tag{B.3}$$

Since, for $1 \leq j \leq k$, it is true that $k-j \geq \log k - \log j$, (B.3) implies (B.1). However we shall show that, in the text of Proposition B.1, one can replace harmlessly, without doing any other change, "the condition (B.1)" by "the condition (B.3)". The new proposition will be called *Proposition B.1 (modified).*

Lemma B.2. *Suppose that the spectrum of a Hamiltonian* H *satisfies the condition (B.1). Then* H *can be decomposed into a direct sum,* $H = \sum_{a\in\mathbb{Z}_+}^{\oplus} H_a$*, so that the spectrum of each summand satisfies the condition (B.3), of course, with modified constants* $C_M > 0$ *and* $\mu > 0$*.*

Proof. Each $j \in \mathbb{N}$ can be written in a unique way as $j = a + 2^k$, with $a, k \in \mathbb{Z}_+$ and $a \leq 2^k - 1$. For a given $a \in \mathbb{Z}_+$, denote by $\kappa(a)$ the smallest non-negative integer such that $a \leq 2^{\kappa(a)} - 1$, and set

$$N(a) := \{a + 2^{\kappa(a)+k-1};\ k \in \mathbb{N}\}\,.$$

According to what we have said,

$$\mathbb{N} = \bigcup_{a\in\mathbb{Z}_+} N(a)$$

is a disjoint union. It induces a decomposition of H, $H = \sum_{a\in\mathbb{Z}_+}^{\oplus} H_a$, so that $\mathrm{Spec}(H_a) = \{E_k;\ k \in N(a)\}$. Furthermore, if $j, k \in \mathbb{N}$ and $j < k$ then

$$C_M \left(\frac{a + 2^{\kappa(a)+k-1}}{a + 2^{\kappa(a)+j-1}}\right)^{\mu} \geq C_M \left(\frac{2^{\kappa(a)+k-1}}{(a+1)2^{\kappa(a)+j-1}}\right)^{\mu} = \frac{C_M}{(a+1)^{\mu}}\ \mathrm{e}^{\mu\log 2\cdot(k-j)}\,.$$

We conclude that if $\mathrm{Spec}(H)$ satisfies (B.1) then $\mathrm{Spec}(H_a)$ satisfies (B.3). □

Corollary B.3. *Proposition B.1 (modified) implies Proposition B.1.*

Proof. Suppose that $\mathrm{Spec}(H)$ satisfies (B.1). Decompose, in accordance with Lemma B.2, $H = \sum_{a\in\mathbb{Z}_+}^{\oplus} H_a$, and apply to each summand Proposition B.1 (modified) getting this way a family of perturbations $V_a(t)$, $a \in \mathbb{Z}_+$ (acting in mutually orthogonal

subspaces). Then the perturbation $V(t) := \sum_{a\in\mathbb{Z}_+}^{\oplus} V_a(t)$ obeys the conclusions of Proposition B.1. □

Construction of the perturbation. Set

$$V(t) := \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} 2\cos([\,\omega^{-1}|E_j-E_k|]t)\,\sqrt{\xi_j\xi_k}\,\langle e_k,\cdot\rangle_{\mathcal{H}}\,e_j$$

where

$$\xi_k := \frac{1}{k\,\log^2(k+1)},\quad k\in\mathbb{N}. \tag{B.4}$$

Here $[x]$ denotes the integer part of $x\in\mathbb{R}$. In other words, the definition of $V(t)$ means that

$$\langle e_j, V(t)e_k\rangle_{\mathcal{H}} = 2\cos([\,\omega^{-1}|E_j-E_k|]t)\,\sqrt{\xi_j\xi_k},\quad j,k\in\mathbb{N}.$$

Furthermore, from the prescription (2.2) one finds that V_{mn}, with $m\neq n$, equals either $\sqrt{\xi_{m_2}\xi_{n_2}}$ or 0, and the former case occurs if and only if $|m_1-n_1| = [\,\omega^{-1}|E_{m_2}-E_{n_2}|]$. For the diagonal entries we have $V_{mm}=2\xi_{m_2}$.

Before proving that $V(t)$ actually fulfills the conclusions of Proposition B.1 (modified) we shall derive some auxiliary results. Nevertheless one can make already now some straightforward observations. First, $V(t)$ is 2π-periodic and the matrix $(\langle e_j, V(t)e_k\rangle_{\mathcal{H}})$ is real and symmetric. Second,

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|\langle e_j,V(t)e_k\rangle_{\mathcal{H}}|^2 \leq 4\left(\sum_{k\in\mathbb{N}}\xi_k\right)^2 < \infty$$

and so $V(t)$ is Hilbert-Schmidt, for each t, and strongly continuous in t. Third, assuming that an index $\eta\in\mathbb{Z}\times\mathbb{N}$ has been chosen, we find that if $V_{n\eta}\neq 0$ and $n_2>\eta_2$ then (again $F\equiv F_\eta$)

$$F_n - F = \omega\{\omega^{-1}(E_{n_2}-E_{\eta_2})\}\quad\text{for } n_1=\eta_1-[\,\omega^{-1}(E_{n_2}-E_{\eta_2})]\,,$$

and

$$F_n - F > E_{n_2}-E_{\eta_2} > 0\quad\text{for } n_1=\eta_1+[\,\omega^{-1}(E_{n_2}-E_{\eta_2})]\,.$$

Here

$$\{x\} := x-[x]\in[\,0,1[$$

is the fractional part of $x\in\mathbb{R}$ (in the text one has to distinguish between the fractional part $\{x\}$ and the sequence $\{x_k\}_k$). Hence

$$\sum_{n\in\mathbb{Z}\times\mathbb{N},\ n_2>\eta_2}\frac{1}{F_n-F}\,|V_{n\eta}|^2 > \frac{\xi_{\eta_2}}{\omega}\sum_{k\in\mathbb{N},\ k>\eta_2}\frac{\xi_k}{\{\omega^{-1}(E_k-E_{\eta_2})\}}\,. \tag{B.5}$$

Let us add an obvious remark that the sub-sum of (B.2), with the summation index being restricted by $n_2\leq\eta_2$, has only finitely many nonzero summands.

In the remainder of this appendix we adopt the point of view of the theory of probability. More precisely, the Lebesgue measure on $[\,0,1\,]$ will be interpreted as a probability measure. This is reflected in the notation, too. We write, for a measurable set $A\subset[\,0,1\,]$, $\mathbb{P}(A)$ instead of $|A|$, and consider the measurable functions on the interval $[\,0,1\,]$ as random variables; here we denote them by the capital letters $X,Y,Z,\ldots$. As usual, $\mathbb{E}(X)$ means the mathematical expectation (mean value).

Denote by χ_N, with $N\in[\,1,+\infty[$, the characteristic function of the interval $]N^{-1},1[$.

Lemma B.4. *Suppose that $M, N \in [1, +\infty[$ and $p, q \in \mathbb{R}$, $1 \leq p \leq q$. Set, for $\zeta \in \mathbb{R}$,*

$$Y(\zeta) := \sum_{k \in \mathbb{Z}} \chi_M(p\zeta - k)\, \frac{1}{p\zeta - k}\,,$$
$$Z(\zeta) := \sum_{k \in \mathbb{Z}} \chi_N(q\zeta - k)\, \frac{1}{q\zeta - k}\,.$$

Then it holds, for the restrictions of Y and Z to the interval $[0,1]$, that

$$|\mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z)| \leq 9M\, \frac{p}{q}\, \log N\,. \tag{B.6}$$

Proof. The verification of (B.6) is based on explicit calculations and rather lengthy but elementary estimates. We only sketch the proof while indicating some intermediate steps.

Observe that the function Y is p^{-1}-periodic and, for each $k \in \mathbb{Z}$, it vanishes on the interval $[\, p^{-1}k, p^{-1}(k + M^{-1})\,]$ and is decreasing on $]p^{-1}(k+M^{-1}), p^{-1}(k+1)[$ from the limit value M to the limit value 1. The integral of Y over the period is $p^{-1}\log M$ and so it is clear that the mathematical expectation $\mathbb{E}(Y)$ is close to $\log M$ provided p is large. More precisely,

$$|\mathbb{E}(Y) - \log M| \leq \frac{1}{p}\, \log M, \quad |\mathbb{E}(Z) - \log N| \leq \frac{1}{q}\, \log N\,. \tag{B.7}$$

Let us consider a bit more general situation and compose Z with a translation,

$$Z_a(\zeta) := Z(\zeta + q^{-1}a) \quad \text{where } a \in \mathbb{R}.$$

This time it holds

$$|\mathbb{E}(Z_a) - \log N| \leq \frac{2}{q}\, \log N\,. \tag{B.8}$$

As a next step we treat the particular case with $p = 1$. We claim that

$$p = 1 \Longrightarrow |\mathbb{E}(YZ_a) - \mathbb{E}(Y)\mathbb{E}(Z_a)| \leq 3\, \frac{M}{q}\, \log N\,. \tag{B.9}$$

Indeed, now we have the precise equality $\mathbb{E}(Y) = \log M$ and so (note that $\log x \leq x/\mathrm{e}$ for $x \geq 1$)

$$|\mathbb{E}(Y)\mathbb{E}(Z_a) - \log(M)\log(N)| \leq \frac{2}{q}\, \log(M)\log(N) < \frac{M}{q}\, \log N\,. \tag{B.10}$$

Furthermore, it holds

$$\mathbb{E}(YZ_a) = \sum_{k(a) \leq k < q - N^{-1} + a} \int_{\max\{M^{-1}, q^{-1}(k+N^{-1}-a)\}}^{\min\{1, q^{-1}(k+1-a)\}} \frac{d\zeta}{\zeta(q\zeta + a - k)}$$

where $k(a) := [M^{-1}q + a]$. It follows that

$$\mathbb{E}(YZ_a) \le \frac{M}{q} \int_{k(a)+N^{-1}-a}^{k(a)+1-a} \frac{d\xi}{\xi + a - k(a)} \quad + \\ \sum_{k(a)<k<q-N^{-1}+a} \frac{1}{k-a} \int_{k+N^{-1}-a}^{k+1-a} \frac{d\xi}{\xi + a - k}$$

and

$$\mathbb{E}(YZ_a) \ge \sum_{k(a)\le k\le q-1+a} \frac{1}{k+1-a} \int_{k+N^{-1}-a}^{k+1-a} \frac{d\xi}{\xi + a - k}\,.$$

Proceeding this way one derives rather straightforwardly that

$$|\mathbb{E}(YZ_a) - \log(M)\log(N)| \le 2\,\frac{M}{q}\,\log N\,. \tag{B.11}$$

The inequalities (B.10) and (B.11) imply (B.9).

The just discussed particular case (B.9) will be useful when verifying the general case. Set

$$J(k) := [\,p^{-1}(k+M^{-1}), p^{-1}(k+1)] \quad \text{for } 0 \le k \le [p]-1\,.$$

We put also $J([p])$ equal to $[\,p^{-1}([p]+M^{-1}), 1\,]$ if $M^{-1} < \{p\}$ and to $\emptyset$ otherwise. Thus we get

$$\begin{aligned} |\mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z)| &= |\mathbb{E}\big(Y(Z-\mathbb{E}(Z))\big)| \\ &\le \sum_{0\le k\le[p]} \left| \int_{J(k)} \frac{1}{p\zeta - k}\, Z(\zeta)\,d\zeta - \log N \int_{J(k)} \frac{d\zeta}{p\zeta - k} \right| \\ &\quad + \left(\sum_{0\le k\le[p]} \int_{J(k)} \frac{d\zeta}{p\zeta - k} \right) |\log N - \mathbb{E}(Z)|\,. \end{aligned} \tag{B.12}$$

According to (B.7), the last term in (B.12) can be estimated from above by

$$([p]+1)\,\frac{1}{p}\,\log M \cdot \frac{1}{q}\,\log N < \frac{M}{q}\,\log N\,. \tag{B.13}$$

On the other hand, for $0 \le k < [p]$,

$$\begin{aligned} \left| \int_{J(k)} \frac{1}{p\zeta - k}\, Z(\zeta)\,d\zeta - \log N \int_{J(k)} \frac{d\zeta}{p\zeta - k} \right| &\le \frac{1}{p} \left| \int_{M^{-1}}^{1} \frac{1}{\xi}\, \tilde{Z}(\xi)\,d\xi - \log(M)\mathbb{E}(\tilde{Z}) \right| \\ &\quad + \frac{1}{p}\,\log(M)\,|\mathbb{E}(\tilde{Z}) - \log N| \\ &\le 4\,\frac{M}{q}\,\log N \end{aligned} \tag{B.14}$$

where we have set $\tilde{Z}(\xi) := Z(p^{-1}\xi + p^{-1}k)$ and then we have applied (B.9) and (B.8) with $\tilde{q} = q/p$ and $\tilde{a} = kq/p$. Quite similarly one estimates

$$\left| \int_{J([p])} \frac{1}{p\zeta - [p]}\, Z(\zeta)\,d\zeta - \log N \int_{J([p])} \frac{d\zeta}{p\zeta - [p]} \right| \le 4\,\frac{M}{q}\,\log N\,. \tag{B.15}$$

Combining (B.12), (B.13), (B.14) and (B.15) we get

$$|\mathbb{E}(YZ) - \mathbb{E}(Y)\mathbb{E}(Z)| \leq ([p]+1)\, 4\, \frac{M}{q}\, \log N + \frac{M}{q}\, \log N \leq 9M\, \frac{p}{q}\, \log N\,,$$

as required. □

Next we will treat a sequence of random variables of the same type as Y but with a particular choice of the parameters M and p. Fix θ, $0 < \theta < 1/2$, and set

$$Y_k(\zeta) := \sum_{j \in \mathbb{Z}_+} \chi_{k^\theta}(h_k\zeta - j)\, \frac{1}{h_k\zeta - j}\,, \quad k \in \mathbb{N}, \tag{B.16}$$

where $\{h_k\}_k$ is a sequence of positive numbers. We assume that $h_k \geq 1$, $\forall k$, and that the sequence obeys the same type of condition as given in (B.3); this is to say that there exist constants $C_M > 0$ and $\mu > 0$ such that

$$j < k \Longrightarrow \frac{h_k}{h_j} \geq C_M\, \mathrm{e}^{\mu(k-j)}\,. \tag{B.17}$$

Let us specialize some estimates to the random variable Y_k. According to (B.7) we have

$$|\mathbb{E}(Y_k) - \theta \log k| \leq \frac{\theta \log k}{h_k}\,. \tag{B.18}$$

Quite similarly, it holds true that

$$|\mathbb{E}(Y_k^2) - k^\theta + 1| \leq \frac{k^\theta - 1}{h_k}\,. \tag{B.19}$$

As a consequence of (B.19) we get

$$\mathbb{E}(Y_k^2) \leq 2\, k^\theta\,. \tag{B.20}$$

Finally, Lemma B.4 jointly with (B.17) tells us that

$$\begin{aligned} j < k \Longrightarrow |\mathbb{E}(Y_jY_k) - \mathbb{E}(Y_j)\mathbb{E}(Y_k)| &\leq 9j^\theta\, \log(k^\theta)\, \frac{h_j}{h_k} \\ &\leq \frac{9\theta}{C_M}\, j^\theta\, \log(k)\, \mathrm{e}^{-\mu(k-j)}\,. \end{aligned} \tag{B.21}$$

Lemma B.5. *Suppose that a sequence $\{h_k\}_{k\in\mathbb{N}}$ satisfies $h_k \geq 1$, $\forall k$, and the condition (B.17). Set*

$$S_N := Y_1 + \cdots + Y_N\,, \quad N \in \mathbb{N}, \tag{B.22}$$

where Y_k has been defined in (B.16). Then

$$\lim_{N\to\infty} \frac{1}{N}\, \big(S_N - \mathbb{E}(S_N)\big) = 0 \tag{B.23}$$

almost everywhere on $[0,1]$.

Proof. This would be a classical text-book result if the random variables Y_k were independent (see §5.1 in [4]). The estimate (B.21) guarantees that, in our case, the random variables are correlated sufficiently weakly. We only sketch the proof. Set

$$Y_k' := Y_k - \mathbb{E}(Y_k), \quad S_N' := S_N - \mathbb{E}(S_N) = Y_1' + \cdots + Y_N' \,.$$

Fix θ' such that $\theta < \theta' < 1/2$. Using (B.20) and (B.21) we estimate

$$\begin{aligned}
\mathbb{E}\big((S_N')^2\big) &= \sum_{1\le k\le N} \mathbb{E}\big((Y_k')^2\big) + 2 \sum_{1\le j<k\le N} \mathbb{E}(Y_j'Y_k') \\
&\le \sum_{1\le k\le N} \mathbb{E}(Y_k^2) + 2 \sum_{1\le j<k\le N} |\mathbb{E}(Y_jY_k) - \mathbb{E}(Y_j)\mathbb{E}(Y_k)| \\
&\le 2 \sum_{1\le k\le N} k^\theta + \frac{18\theta}{C_M} \sum_{1\le j\le N} j^\theta \log N \sum_{s\in\mathbb{N}} e^{-\mu s} \\
&\le C_I\, N^{1+\theta'}
\end{aligned}$$

where $C_I > 0$ is a constant. According to the Chebyshev's inequality we have, for $\varepsilon > 0$,

$$\mathbb{P}(|S_N'| > N\varepsilon) \le \frac{C_I\, N^{1+\theta'}}{N^2\varepsilon^2} = \frac{C_I}{\varepsilon^2 N^{1-\theta'}} \,,$$

and so

$$\sum_{N\in\mathbb{N}} \mathbb{P}(|S_N'| > N^2\varepsilon) < \infty \,.$$

By the Borel-Cantelli lemma, $\limsup N^{-2}|S_{N^2}'| \le \varepsilon$ holds true for all $\varepsilon > 0$ and for almost all $\zeta \in [0,1]$. One deduces from this, by a standard argument, that

$$\lim_{N\to\infty} \frac{1}{N^2} S_{N^2}' \equiv \lim_{N\to\infty} \frac{1}{N^2}\big(S_{N^2} - \mathbb{E}(S_{N^2})\big) = 0 \tag{B.24}$$

almost everywhere on $[0,1]$. To pass from (B.24) to (B.23) we introduce the random variables

$$D_N := \max_{N^2<M<(N+1)^2} |S_{N^2}' - S_M'| \,.$$

Using basically the same estimates as before one shows that

$$\sum_{N\in\mathbb{N}} \mathbb{P}(D_N > N^2\varepsilon) \le \sum_{N\in\mathbb{N}} \frac{C_{II}}{\varepsilon^2 N^{2(1-\theta')}} < \infty$$

and so $\lim N^{-2} D_N = 0$ almost everywhere on $[0,1]$. To complete the proof it suffices to observe that, for $N^2 < M < (N+1)^2$,

$$\frac{|S_M'|}{M} \le \frac{|S_{N^2}'| + D_N}{N^2} \,. \qquad \square$$

Lemma B.6. *Suppose that $\{h_k\}_{k\in\mathbb{N}}$, a sequence of positive numbers, satisfies the condition (B.17), and ξ_k is the same as in (B.4). Then*

$$\sum_{k\in\mathbb{N}} \frac{\xi_k}{\{\zeta h_k\}} = +\infty \tag{B.25}$$

for almost all $\zeta > 0$.

Proof. Clearly it suffices to show that (B.25) is valid for almost all ζ from an arbitrary bounded interval $[\,0, z\,]$, $z > 0$. Having observed that the condition (B.17) is invariant with respect to the scaling of h_k we can restrict ourselves to $\zeta \in [\,0, 1\,]$. Furthermore, the conclusion of the lemma is not influenced by omitting several first numbers of the sequence $\{h_k\}_k$. This is why we can assume that the assumptions of Lemma B.5 are satisfied.

Set

$$X_k(\zeta) := 1/\{\zeta h_k\}, \quad k \in \mathbb{N}.$$

Note that Y_k (cf. (B.16)) is nothing but the cutoff of the function X_k obtained by annulling the values which exceed the level k^θ; hence $X_k \geq Y_k$. The symbol S_N retains its meaning from (B.22). We have

$$\begin{aligned}\sum_{k\in\mathbb{N}} \xi_k X_k &\geq \sum_{k\in\mathbb{N}} \xi_k Y_k \\ &= \sum_{k\in\mathbb{N}} (\xi_k - \xi_{k+1})(S_k - \mathbb{E}(S_k)) + \sum_{k\in\mathbb{N}} (\xi_k - \xi_{k+1})\mathbb{E}(S_k)\,.\end{aligned}$$

It is elementary to derive the estimate

$$\frac{1}{(1+k)^2 \log^2(2+k)} \leq \xi_k - \xi_{k+1} \leq \frac{3}{k^2 \log^2(1+k)}\,.$$

Hence $\sum_k k(\xi_k - \xi_{k+1}) < \infty$. Since, by Lemma B.5, the sequence $\{k^{-1}(S_k - \mathbb{E}(S_k))\}_{k\in\mathbb{N}}$ is bounded almost everywhere we find that the sum

$$\sum_{k\in\mathbb{N}} (\xi_k - \xi_{k+1})(S_k - \mathbb{E}(S_k))$$

converges for almost all ζ. To finish the proof we need to estimate $\mathbb{E}(S_k)$. The inequality (B.18) and the fact that $\lim h_k = +\infty$ imply that there exists $k_\star \in \mathbb{N}$ such that

$$\mathbb{E}(Y_k) \geq \frac{1}{2}\,\theta \log k, \quad \text{for } k \geq k_\star\,.$$

Consequently, if $k \geq k_\star$, then

$$\mathbb{E}(S_k) \geq \frac{1}{2}\,\theta \sum_{k_\star \leq j \leq k} \log j \geq C_{III}\,(1+k)\log(2+k)$$

where $C_{III} > 0$ is a constant. Hence

$$\sum_{k\in\mathbb{N}} (\xi_k - \xi_{k+1})\mathbb{E}(S_k) \geq C_{III} \sum_{k\geq k_\star} \frac{1}{(1+k)\log(2+k)} = +\infty\,.$$

This completes the proof. $\square$

Proof of Proposition B.1 (modified). In virtue of the inequality (B.5) and the remark following it, it is sufficient to apply Lemma B.6 to the sequence $\{h_k\}_{k\in\mathbb{N}}$ defined by

$$h_k := E_k - E_{\eta_2}, \quad k \in \mathbb{N},\ k > \eta_2\,,$$

(in fact, we treat a countable family of such sequences labeled by $\eta_2 \in \mathbb{N}$). Observe that if $\eta_2 < j < k$ then $h_k/h_j \geq E_k/E_j$ and so (B.3) implies (B.17). Hence the assumption of Lemma B.6 is indeed satisfied. $\square$

Appendix C. Lipschitz functions

Here we present some auxiliary results concerning Lipschitz functions which are quite straightforward to verify but are not mentioned in [23], our main source on this subject. Moreover, in view of applications we are interested in, we allow the target space to be generally a Banach algebra (sometimes only a Banach space) rather than $\mathbb{C}$. In fact, this doesn't cause any essential complication – one has just to be careful about the order of multipliers in all expressions. The notation in this appendix is autonomous, particularly the symbols f, g, P etc. have different meaning in the main text of the paper.

Definition C.1. *Suppose that $\Pi \subset \mathbb{R}^n$ is a closed set and $\mathcal{A}$ is a Banach algebra (or just a Banach space). A function f defined on Π and with values in $\mathcal{A}$ belongs to the Lipschitz class $\mathrm{Lip}(\ell+\varepsilon,\Pi)$, with $\ell \in \mathbb{Z}_+$ and $0 < \varepsilon \leq 1$, if and only if there exists a family of functions $\{f^{(\nu)};\ \nu \in \mathbb{Z}_+^n,\ |\nu| \leq \ell\}$, with $f^{(0)} \equiv f$, and a constant $M > 0$ such that, for all $\nu \in \mathbb{Z}_+^n$, $|\nu| \leq \ell$, it holds true that*

$$\begin{aligned}
&|f^{(\nu)}(x)| \leq M, \quad \textit{for all } x \in \Pi,\\
&|f^{(\nu)}(x) - \partial_x^\nu P(x,y)| \leq M\,|x-y|^{\ell+\varepsilon-|\nu|}, \quad \textit{for all } x, y \in \Pi,
\end{aligned}$$

where

$$P(x,y) := \sum_{\mu,\ |\mu|\leq\ell} f^{(\mu)}(y)\,\frac{(x-y)^\mu}{\mu!}\,.$$

The smallest constant M with this property is called the Lipschitz norm $M(f)$.

As one can guess, we have denoted the norm in $\mathcal{A}$ by $|\cdot|$. If not specified otherwise, the multiindices $\mu, \nu, \ldots$ are assumed to belong to $\mathbb{Z}_+^n$. We use the partial ordering on $\mathbb{Z}_+^n$: $\mu \leq \nu$ means that $\mu_j \leq \nu_j$ for all j, $1 \leq j \leq n$. Set

$$R_\nu(x,y) := f^{(\nu)}(x) - \partial_x^\nu P(x,y), \quad \nu \in \mathbb{Z}_+^n,\ |\nu| \leq \ell.$$

If necessary, the dependence of $P(x,y)$ or $R(x,y)$ on f will be distinguished by a superscript. A detailed proof of the following basic theorem is given in [23].

Theorem C.2 (Whitney Extension Theorem). *There exists a continuous mapping $\mathcal{E} : \mathrm{Lip}(\ell+\varepsilon,\Pi) \to \mathrm{Lip}(\ell+\varepsilon,\mathbb{R}^n)$ such that $\mathcal{E}(f)$ is an extension of f for all $f \in \mathrm{Lip}(\ell+\varepsilon,\Pi)$. The norm of $\mathcal{E}$ has a bound independent of Π.*

We shall frequently use the observation that $f \in \mathrm{Lip}(\ell+\varepsilon,\mathbb{R}^n)$ if and only if $f \in C^\ell(\mathbb{R}^n)$, all derivatives of f up to the order ℓ are uniformly bounded on $\mathbb{R}^n$,

and $\partial_x^\nu f \in \mathrm{Lip}(\varepsilon, \mathbb{R}^n)$ for all ν, $|\nu| = \ell$. This claim still holds true when replacing $\mathbb{R}^n$ by a closed convex subset Π of dimension n. Clearly,

$$f \in \mathrm{Lip}(\ell+\varepsilon, \Pi) \Longrightarrow f^{(\nu)} \in \mathrm{Lip}(\ell-|\nu|+\varepsilon, \Pi), \quad \text{for all } \nu,\ |\nu| \le \ell. \tag{C.1}$$

The family of functions corresponding to $f^{(\nu)}$ is $\{f^{(\nu+\mu)}\}_{0\le|\mu|\le\ell-|\nu|}$. The extension operator has the property that $f^{(\nu)}(x) = \partial_x^\nu \mathcal{E}(f)(x)$ holds true for all $x \in \Pi$ and all $\nu \in \mathbb{Z}_+^n$, $|\nu| \le \ell$. The following proposition is quite easy to verify.

Proposition C.3. *Suppose that $\ell \ge 1$ and Π is bounded. If $f \in \mathrm{Lip}(\ell+\varepsilon, \Pi)$ then $f \in \mathrm{Lip}(\ell'+\varepsilon', \Pi)$ for all ℓ', $0 \le \ell' < \ell$, and any ε', $0 < \varepsilon' \le 1$. The embedding mapping $\mathcal{I}_{\ell,\ell'} : \mathrm{Lip}(\ell+\varepsilon, \Pi) \to \mathrm{Lip}(\ell'+\varepsilon', \Pi)$, sending the family $\{f^{(\nu)}\}_{|\nu|\le\ell}$ to $\{f^{(\nu)}\}_{|\nu|\le\ell'}$, is bounded.*

In the following two propositions we shall need the structure of algebra on $\mathcal{A}$.

Proposition C.4. *Suppose that Π is bounded and both f and g belong to $\mathrm{Lip}(\ell+\varepsilon, \Pi)$. Then $fg \in \mathrm{Lip}(\ell+\varepsilon, \Pi)$.*

Proof. Set $h = fg$ and, more generally,

$$h^{(\nu)} = \sum_{\mu,\ \mu\le\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, f^{(\mu)} g^{(\nu-\mu)}\,.$$

Then

$$P^h(x,y) = P^f(x,y)\, P^g(x,y) - \Psi(x,y)$$

where

$$\Psi(x,y) := \sum_{\substack{\mu,\nu \\ |\mu|\le\ell,\ |\nu|\le\ell,\ |\mu+\nu|\ge\ell+1}} f^{(\mu)}(y)\, g^{(\nu)}(y)\, \frac{(x-y)^{\mu+\nu}}{\mu!\,\nu!}\,. \tag{C.2}$$

It follows that

$$\begin{aligned} h^{(\nu)}(x) - \partial_x^\nu P^h(x,y) &= \\ \sum_{\mu,\ \mu\le\nu} \frac{\nu!}{\mu!(\nu-\mu)!} \big(R_\mu^f(x,y)\, \partial_x^{\nu-\mu} P^g(x,y) &+ f^{(\mu)}(x)\, R_{\nu-\mu}^g(x,y)\big) \\ &+ \partial_x^\nu \Psi(x,y)\,. \end{aligned}$$

We conclude that $|h^{(\nu)}(x)| = O(1)$ and $|R_\nu^h(x,y)| = O(|x-y|^{\ell+\varepsilon-|\nu|})$. $\square$

Proposition C.5. *Suppose that Π is bounded, $f \in \mathrm{Lip}(\ell+\varepsilon, \Pi)$, $f(x)^{-1}$ exists in $\mathcal{A}$ for all $x \in \Pi$, and $|f(x)^{-1}|$ is uniformly bounded on Π. Then $g \in \mathrm{Lip}(\ell+\varepsilon, \Pi)$ where $g(x) = f(x)^{-1}$.*

If, in addition, the diameter $\operatorname{diam}\Pi \le 1$ and $|f(x)^{-1}| \le \kappa$ for all $x \in \Pi$ then

$$M(g) \le C_L\, M(f)^{\ell+1} \kappa^{\ell+2}$$

where $C_L \equiv C_L(n,\ell)$ is a constant.

Proof. We can assume from the beginning that $\operatorname{diam}\Pi \le 1$ and, by rescaling f, that $|f(x)^{-1}| \le 1$ on Π, i.e., $\kappa = 1$ (the norm $M(\cdot)$ is homogeneous). Then we have

$$|f(x)| \ge |f(x)^{-1}||f(x)| \ge 1$$

and so $M(f) \geq 1$.

We define successively, for $1 \leq |\nu| \leq \ell$,

$$g^{(\nu)} = -g\bigg(\sum_{\mu,\ \mu<\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, f^{(\nu-\mu)} g^{(\mu)} \bigg). \tag{C.3}$$

This means that the identity

$$\sum_{\mu,\ \mu\leq\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, f^{(\nu-\mu)}(x)\, g^{(\mu)}(x) = \delta_{\nu 0} \tag{C.4}$$

is valid for all $\nu \in \mathbb{Z}_+^n$, $|\nu| \leq \ell$, and all $x \in \Pi$.

Clearly, all $g^{(\nu)}$ are bounded. We know, by the assumption, that $|g(x)| \leq 1$, and we claim that

$$|g^{(\nu)}(x)| \leq \big(|\nu|\, M(f)\big)^{|\nu|} \quad \text{for } 1 \leq |\nu| \leq \ell,\ \forall x \in \Pi. \tag{C.5}$$

To see (C.5) we proceed by induction $|\nu|$ using the formula (C.3),

$$\begin{aligned} |g^{(\nu)}(x)| &\leq \sum_{\mu,\ \mu<\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, M(f)\, \big(|\mu|\, M(f)\big)^{|\mu|} \\ &\leq M(f)^{|\nu|} \sum_{\mu,\ \mu\leq\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, (|\nu|-1)^{|\mu|} \\ &= \big(|\nu|\, M(f)\big)^{|\nu|}. \end{aligned}$$

With the aid of (C.4) one finds readily that

$$P^f(x,y)\, P^g(x,y) = 1 + \Psi(x,y) \tag{C.6}$$

where Ψ is the same as in (C.2). Differentiating (C.6) and subtracting (C.4) from the result one arrives at ($0 \leq |\nu| \leq \ell$)

$$\sum_{\mu,\ \mu\leq\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, \big(\partial_x^{\nu-\mu} P^f(x,y) \cdot \partial_x^{\mu} P^g(x,y) - f^{(\nu-\mu)}(x)\, g^{(\mu)}(x)\big) = \partial_x^{\nu}\, \Psi(x,y)\,. \tag{C.7}$$

More conveniently, let us rewrite (C.7) as a recurrence formula,

$$\begin{aligned} -f(x)\, R_\nu^g(x,y) = &\sum_{\mu,\ \mu<\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, f^{(\nu-\mu)}(x)\, R_\mu^g(x,y) \\ &+ \sum_{\mu,\ \mu\leq\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\, R_{\nu-\mu}^f(x,y)\, \partial_x^{\mu} P^g(x,y) \\ &+ \partial_x^{\nu}\, \Psi(x,y)\,. \end{aligned} \tag{C.8}$$

Using (C.5) one finds easily the required bounds for a part of the RHS of (C.8). Namely,

$$\begin{aligned} |\partial_x^{\nu} P^g(x,y)| &\leq \sum_{\mu,\ |\mu|\leq\ell-|\nu|} \big(|\nu+\mu|\, M(f)\big)^{|\nu+\mu|}\, \frac{|x-y|^{|\mu|}}{\mu!} \\ &\leq \sum_{\mu\in\mathbb{Z}_+^n} \frac{1}{\mu!}\, \big(\ell M(f)\big)^{\ell} \end{aligned}$$

and so ($|x-y|\leq 1$)

$$|R^f_{\nu-\mu}(x,y)\,\partial_x^\mu\,P^g(x,y)|\leq \mathrm{e}^n\ell^\ell M(f)^{\ell+1}|x-y|^{\ell+\varepsilon-|\nu|}\,. \tag{C.9}$$

Furthermore,

$$\begin{aligned}|\partial_x^\nu\Psi(x,y)|\leq &\sum_{\substack{|\sigma|\leq\ell,\ |\mu|\leq\ell\\ |\sigma+\mu|\geq\ell+1,\ \sigma+\mu\geq\nu}}|f^{(\sigma)}(y)|\,|g^{(\mu)}(y)|\,\frac{(\sigma+\mu)!}{\sigma!\,\mu!\,(\sigma+\mu-\nu)!}\,|x-y|^{|\sigma+\mu|-|\nu|}\\ &\leq \text{const}\ M(f)^{\ell+1}\,|x-y|^{\ell+1-|\nu|}\,.\end{aligned} \tag{C.10}$$

So it remains to estimate, in the same manner, the first sum on the RHS of (C.8). The three estimates ((C.9), (C.10) and the one still lacking) should amount in the existence of constants $c_\nu>0$ (depending also on n and ℓ, $0\leq|\nu|\leq\ell$) such that

$$|R^g_\nu(x,y)|\leq c_\nu\,M(f)^{\ell+1}|x-y|^{\ell+\varepsilon-|\nu|}\,. \tag{C.11}$$

To prove the proposition we shall proceed by induction in ℓ. The case $\ell=0$ is obvious for $|g(x)|\leq 1\leq M(f)$ and

$$|R^g(x,y)|=|-f(x)^{-1}R^f(x,y)\,f(y)^{-1}|\leq M(f)\,|x-y|^\varepsilon\,;$$

hence $M(g)\leq M(f)$. Suppose now that $\ell\geq 1$ and the proposition is valid for all ℓ', $0\leq\ell'<\ell$. Write, for $\mu<\nu$,

$$R^g_\mu(x,y)=R'_\mu(x,y)+\sum_{\substack{\sigma\\ \ell'-|\mu|<\sigma\leq\ell-|\mu|}}g^{(\mu+\sigma)}(y)\,\frac{(x-y)^\sigma}{\sigma!}$$

where $\ell'=\ell-|\nu-\mu|$ and R'_μ is the rest function related to $\mathcal{I}_{\ell,\ell'}(g)\in\mathrm{Lip}(\ell'+1,\Pi)$. Note that $|\mu|\leq\ell'<\ell$. By the induction hypothesis and by Proposition C.3, we have ($\ell'+1-|\mu|\geq\ell+\varepsilon-|\nu|$)

$$\begin{aligned}|R'_\mu(x,y)|&\leq C_L(n,\ell')\,M(\mathcal{I}_{\ell,\ell'}(f))^{\ell'+1}|x-y|^{\ell'+1-|\mu|}\\ &\leq C_L(n,\ell')\,\|\mathcal{I}_{\ell,\ell'}\|^\ell\,M(f)^\ell|x-y|^{\ell+\varepsilon-|\nu|}\,.\end{aligned}$$

In addition, for $|\sigma|>\ell'-|\mu|$ and $|\mu+\sigma|\leq\ell$,

$$|g^{(\mu+\sigma)}(y)\,(x-y)^\sigma|\leq\big(\ell\,M(f)\big)^\ell|x-y|^{\ell'+1-|\mu|}\leq\big(\ell\,M(f)\big)^\ell|x-y|^{\ell+\varepsilon-|\nu|}\,.$$

We conclude that

$$|f^{(\nu-\mu)}(x)\,R^g_\mu(x,y)|\leq\text{const}\,M(f)^{\ell+1}|x-y|^{\ell+\varepsilon-|\nu|}\,. \tag{C.12}$$

The formula (C.8) and the bounds (C.9), (C.10) and (C.12) prove the validity of (C.11). □

The last auxiliary result concerns the composition of functions. This time $\mathcal{A}$ is a Banach space.

Proposition C.6. *Suppose that* $g : \mathbb{R}^n \to \mathcal{A}$ *belongs to* $\mathrm{Lip}(\ell + \varepsilon, \mathbb{R}^n)$.

(i) *If* $\ell = 0$ *and* $f : \Pi \to \mathbb{R}^n$ *belongs to* $\mathrm{Lip}(1, \Pi)$ *then* $g \circ f \in \mathrm{Lip}(\varepsilon, \Pi)$.

(ii) *If* $\ell \geq 1$ *and* $f : \mathbb{R}^m \to \mathbb{R}^n$ *belongs to* $\mathrm{Lip}(\ell+\varepsilon, \mathbb{R}^m)$ *then* $g \circ f \in \mathrm{Lip}(\ell+\varepsilon, \Pi)$ *for any compact set* $\Pi \subset \mathbb{R}^m$.

Proof. (i) This is obvious from the estimates $|g \circ f(x)| \leq M(g)$ and

$$|g \circ f(x) - g \circ f(y)| \leq M(g)\, |f(x) - f(y)|^\varepsilon \leq M(g)\, M(f)^\varepsilon |x - y|^\varepsilon .$$

(ii) We can restrict ourselves to the case when $\Pi = \bar{\mathcal{U}}$ where $\mathcal{U} \subset \mathbb{R}^m$ is an open, convex and bounded set. Write f as an n-tuple of functions: $f = (f_1, \ldots, f_n)$, $f_j : \mathbb{R}^m \to \mathbb{R}$. Clearly $g \circ f \in C^\ell(\Pi)$, Π is compact, and thus we have to show only that $\partial_x^\nu\, g \circ f \in \mathrm{Lip}(\varepsilon, \Pi)$ for all ν, $|\nu| = \ell$. However, $\partial_x^\nu\, g \circ f$ is a polynomial in $\partial_x^\mu f_j$, $1 \leq j \leq n$ and $|\mu| \leq \ell$, and in $(\partial_y^\mu g) \circ f$, $|\mu| \leq \ell$. This means that, when applying an obvious modification of Proposition C.4 (here we multiply scalar functions by vector-valued functions), it suffices to verify that all the multipliers belong to $\mathrm{Lip}(\varepsilon, \Pi)$. By Proposition C.3, $\partial_x^\mu f_j \in \mathrm{Lip}(\varepsilon, \Pi)$ and $f \in \mathrm{Lip}(1, \Pi)$. Furthermore, from the already proven part (i) we conclude that $(\partial_y^\mu g) \circ f \in \mathrm{Lip}(\varepsilon, \Pi)$. This completes the proof. □

Acknowledgements. P. S. wishes to thank his hosts at Centre de Physique Théorique in Marseille and at Université de Toulon et du Var for their hospitality. The support from the grant 202/96/0218 of Czech Grant Agency is also gratefully acknowledged.

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